Small Body Precision Landing via Convex Model Predictive Control

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Spacecraft operation in proximity to small bodies requires advanced guidance, navigation and control (GN&C) protocols that are able to operate and land autonomously. To achieve a successful landing, the on-board GN&C algorithm must reliably land the spacecraft near the targeted surface location, while ensuring a low velocity at touchdown. In this paper, the soft landing problem is reformulated as a convex optimization problem, and Model Predictive Control (MPC) is used to handle both the optimization process and system constraints in a unified framework. We present a new collision avoidance method; an optimal separating hyperplane coupled with a projection theorem argument to define auxiliary set points used in the MPC formulation. We show that these auxiliary set points converge to the desired target state, and thus the spacecraft will reach its goal safely. Numerical simulation results demonstrate the performance.

I. Introduction

The problem of spacecraft trajectory optimization is elegantly stated by Conway as “the determination of a trajectory for a spacecraft that satisfies specified initial and terminal conditions, that is conducting a required mission, while minimizing some quantity of importance.”. Traditionally, this is taken in the context of interplanetary transfer orbits and orbital maneuvers in Earth orbit, and has largely been guided by optimal control with the goal of using the least amount of fuel for a given maneuver. More recently, small body operations have received similar attention. A small body may be a comet, asteroid or small planetoid that exhibits a highly non-uniform gravitational field as compared to larger, more uniform bodies.

Spacecraft operations in proximity to small bodies require advanced guidance, navigation and control (GN&C) protocols. The need for autonomy is a result of the great distances between small bodies and the earth, which causes delays in sending commands to the spacecraft on the order of minutes.

Previous missions that have attempted landings have used extensive observations of the body in question, such as ESA’s Rosetta mission, JAXA’s Hayabusa, and recently NASA’s Osiris-REx. Significant work has been done to characterize the environments around small bodies and describe their dynamic behavior, and future missions that would seek to study or mine asteroids and comets may wish to perform precision landings without as much lead time. To achieve mission success, the on-board GN&C algorithm must therefore autonomously and reliably land the spacecraft near the targeted surface location, while ensuring a low velocity at touchdown and a collision free trajectory. Such is the focus of the present paper.

Throughout this work, the soft landing problem is split into two phases, called the circumnavigation phase and the landing phase. This allows for different constraints to be applied at different times in the mission, which leads to improved performance. The circumnavigation phase, when the spacecraft must follow a collision free trajectory, presents the biggest challenge. The problem formulation is non-convex in its most basic form. We thus devote some effort to convexifying the problem; proposing a new problem that can be solved to global optimality and whose solutions are feasible ones for the original problem. To this end, a novel optimal separating hyperplane constraint is imposed during the circumnavigation phase that ensures spacecraft will not collide with the surface of the asteroid. We use a projection theorem argument in concert to define auxiliary target states that are guaranteed to be feasible points based on the convexification procedure. The result is a sequence of finite horizon optimal control problems that ensure the spacecraft achieves mission success.

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A. Notation

Throughout the paper, we use regular font, lower case letters to refer to scalars ($\alpha \in \mathbb{R}$). Lowercase, bold-face letters denote vectors, whose size is either explicitly stated or is clear from context. Uppercase variables will refer to matrices and sets, i.e. $A \in \mathbb{R}^{6 \times 6}$, with context providing the distinction. The space of $n$ dimensional symmetric positive definite matrices is written as $\mathbb{S}_{++}^n$. We write the cross product operator as a skew symmetric matrix, $\omega \times (\cdot) = \omega \times (\cdot)$. A sequence of vectors is denoted by $\{x_i\}_{i=k}^{k+N}$ and refers to the sequence $\{x_k, x_{k+1}, \ldots, x_{k+N}\}$. All norms and inner products are the standard Euclidean ones, unless specified otherwise.

B. Previous Works

It has been said that “the great watershed in optimization is not between linearity and nonlinearity, but convexity and nonconvexity”.\textsuperscript{15} In line with this view, the convexification of a finite horizon optimal control problem is a trend that has emerged in the recent extensive work on small body proximity control.\textsuperscript{4, 5, 7, 10, 16–20} In these works, non-convexities that are inherent to the small body soft-landing problem are dealt with by restating the problem in such a way that the dynamics and constraints are convex functions of the spacecraft’s state vector and control input.

The theory of lossless convexification, first presented in Ref. 16 and extended in Refs. 18–20, establishes convexified problems that are equivalent to the original problem. This is a promising area of research, but the equivalency between the convexified problem and its original counterpart is difficult to establish and the arguments are specific to the dynamics of the problem. These works do not consider circumnavigation of the asteroid, and focus solely on the landing scenario.

Other methods give a convexified problem that is an approximation to the original problem.\textsuperscript{4, 5, 17} In these works, Refs. 4 and 5 use a reference governor to update the reference point in an LQR setting, then explore convex and nonlinear Model Predictive Control approaches using a rotating hyperplane method. The rate of rotation of the hyperplanes is a design parameter that can be difficult to establish and is problem-specific. In Ref. 17, a method is presented whereby non-convex constraints are linearized, and a solution is obtained iteratively. One of the downsides of both of these approaches is that convergence of to the target point is often difficult to establish \textit{a priori}. A certain degree of conservativeness is also inherent to these methods. For example, both work by replacing a concave constraint – common in obstacle avoidance problems – with an affine approximation.

This research presents a new flavor of convexification for the small-body soft landing problem that leverages the geometry of the problem and the nature of finite horizon optimal control problems.

C. Model Predictive Control

Model predictive control (MPC) is a discrete time control strategy whereby a finite horizon optimal control problem is solved at a given time using predicted system behavior. Posed as an online optimization, MPC takes into account the system dynamics, current state, control and state constraints, and control objectives. The result is a sequence of optimal inputs, equal in length to the finite horizon. The first of these controls is applied to the system as the best current action, and the entire procedure is repeated, using a horizon receded by one time step.\textsuperscript{5–8, 19, 21} See Ref. 22 for a survey of MPC and its applications in the aerospace domain.

As computational power for a given size increases rapidly, so too does our ability to solve large optimization problems on-line and in real-time. This means that systems with typically fast dynamics – a common thread in aerospace – can benefit from methods such as MPC to achieve a higher degree of autonomy, reduced fuel consumption, and better performance.

The desire to decrease fuel consumption and ensure spacecraft safety during the entire mission leads to the formulation of an optimal control problem that is subject to specific constraints; such as collision avoidance with an object, thruster saturation limits, \textit{etc}. Model predictive control is able to handle both the optimization process and system constraints in a unified framework, and hence is a desirable methodology for the constrained soft landing problem. Moreover, MPC allows for real-time implementation using currently available hardware. Custom solvers that exploit the structure of specific problems have been shown to perform well and are maturing rapidly.\textsuperscript{23, 24}
II. Models & Assumptions

A. Gravity Model

One of the main differences associated with small body operations compared to powered planetary descent is the handling of the gravitational field. Due to the irregular shapes, spatially varying and generally unknown density of a small body, the gravity field is complex and uncertain.

There are several models of the gravitational field near small bodies, such as the spherical harmonics, spherical Bessel function and polyhedral models, among others.\textsuperscript{25–27} These models are mathematically more complex than the standard Newtonian inverse-squared method, and introduce a higher degree of nonlinearity in our system. For applications that require real-time on-board computation, a low order estimate of the local gravity coupled with a controller that is robust to uncertainty and disturbances is a preferred option. This reduces on-board computation without necessarily compromising performance, since the specific accelerations from gravity on the surface of medium-sized small bodies are on the order of millimeters per second per second.\textsuperscript{28} These accelerations are assumed to be much smaller than those available from on-board thrusters. As a result, the gravity term can effectively be handled with a low order model, where model mismatch is handled through feedback.

By continually solving open-loop optimal control problems starting from the current state, the MPC framework implicitly provides this feedback. As a result, MPC may preclude higher order gravity models from being necessary to develop good guidance and control techniques.

In this work, we will employ the spherical harmonics gravity model as the “true” model. An assumption in using this model is that all mass elements of the body lie inside the radius of mass distribution, called the Brillouin sphere. Outside of the boundary defined by the Brillouin sphere, the spherical harmonics gravity model is the solution of Laplace’s equation and can model the true gravity well.\textsuperscript{29} Inside this boundary, the spherical harmonics solution can diverge and may not be reliable. For highly irregularly shaped asteroids, this may require additional gravity models that are more accurate, such as the spherical Bessel function model or a polyhedral fit.

The spherical harmonics expansion for the gravity field about a small body is given by

\begin{equation}
  g(r) = -\frac{\mu}{r_0} \sum_{m=0}^{\infty} \sum_{n=0}^{m} c_{n,m} v_{n,m} + s_{n,m} w_{n,m},
\end{equation}

where $\mu$ is the asteroids gravitational parameter, $r_0$ is a reference radius, $c_{n,m}$ and $s_{n,m}$ are Stokes coefficients, and $v_{n,m}$ and $w_{n,m}$ are functions of position calculated from the recursive scheme in Ref. 14. The acceleration due to gravity is

\begin{equation}
  f(r) = -\nabla g(r).
\end{equation}

For this work, data obtained for the asteroid 433 Eros was used.\textsuperscript{28,30}

B. Asteroid Rotation Model

We assume that the asteroid is rotating at a constant rate about its principal axis of inertia. This is a reasonable assumption for small-bodies, since the general spin state of a small body asymptotically reaches its lowest energy state; rotation about the principal axis of largest inertia.\textsuperscript{31} The rotation rate for 433 Eros is nearly constant at 1639.38816 degrees per day.\textsuperscript{28} Since the asteroid is rotating relative to the Earth-centered inertial reference frame, we will do our analysis in a reference frame rotating at this rate.

C. Spacecraft Dynamics

Let $\mathcal{H}$ be a rotating Hill frame centered at the center of mass whose $z$-axis is aligned with the spin axis of the asteroid. Define the relative position and velocity of the spacecraft in the frame $\mathcal{H}$ at time $t$ by $r_t = [x(t) \ y(t) \ z(t)]^T$ and $v_t = [\dot{x}(t) \ \dot{y}(t) \ \dot{z}(t)]^T$, respectively. We then model the spacecraft relative dynamics in the inertial frame as

\begin{equation}
  \ddot{r}_t + \omega^*_z r_t + 2 \omega^*_y \dot{r}_t + \omega^*_z \omega^*_z r_t = f_t(r_t) + u_t,
\end{equation}
where \( \mathbf{f}_t = [f_x(t) \, f_y(t) \, f_z(t)]^T \) is the net force per unit mass experienced by the spacecraft, \( \mathbf{\omega}_t \) is the rotational rate of the asteroid and \( \mathbf{u}_t = [u_x(t) \, u_y(t) \, u_z(t)]^T \) is the control input vector per unit mass.\(^{32} \) We assume that the spacecraft is equipped with six ideal thrusters in three orthogonal directions, so that both negative and positive control inputs can be realized in all directions. These dynamics can be written in state space form by letting \( \mathbf{x}_t = [r^T \, \mathbf{v}^T]^T \). Dropping the argument of time, this gives,

\[
\dot{\mathbf{x}} = \begin{bmatrix}
0 & I_3 \\
-\mathbf{\omega} \times & -2\mathbf{\omega} \times
\end{bmatrix} \mathbf{x} + \begin{bmatrix}
0 \\
I_3
\end{bmatrix} \mathbf{u} + \begin{bmatrix}
0 \\
I_3
\end{bmatrix} \mathbf{f}(\mathbf{r})
\]

\[
\dot{\mathbf{x}} = \mathbf{A}_c \mathbf{x} + \mathbf{B}_c \mathbf{u} + \mathbf{E}_c \mathbf{f}(\mathbf{r})
\]

(4)

where \( (\mathbf{A}_c, \mathbf{B}_c, \mathbf{E}_c) \) denote the continuous time matrices.

For use in the MPC architecture, we need to linearize and discretize the continuous time system. The only nonlinear term in (4) is the gravity term, \( \mathbf{f}(\mathbf{r}) \). To deal with this, a few different approaches have been proposed: a direct linearization of (2) is used Ref. 5; a Taylor series expansion is used in Ref. 7 that places the linearized term in the \( \mathbf{A} \) matrix and treats higher order terms as disturbances. In this work, we propose a new method that takes advantage of MPC.

Once a sequence of optimal states has been solved for using the MPC scheme, these points are used to obtain an estimate of the specific gravity for the next solution. Say, at discrete time \( k \), once a sequence of optimal states has been solved for using the MPC scheme, these points are used to obtain an estimate of the specific gravity for the next solution. Say, at discrete time \( k \), we solve a finite horizon MPC problem and obtain the sequence of states \( \{\mathbf{x}_t\}_{i=k}^{k+N} \). At time \( k+1 \), the accelerations due to gravity are then estimated according to \( \mathbf{f}(\mathbf{r}_i) \), where \( \mathbf{r}_i \) is the first three elements of the state vector \( \mathbf{x}_i \), for all prediction times \( i = k+1, \ldots, (k+1) + N \). Since the gravity field changes continuously and is bounded in the region considered, this approximation yields good results.

Next, an exact discretization scheme was used to obtain an LTV realization of the spacecraft dynamics. The result is a system of the form

\[
\mathbf{x}_k = A_d \mathbf{x}_{k-1} + B_d (\mathbf{u}_{k-1} + \mathbf{f}(\mathbf{x}_{k-1}))
\]

(5)

which serves as our prediction model.

### III. Problem Statement

The mission objective is to land a spacecraft safely on the surface of a rotating small body with a small terminal relative velocity (a “soft” landing). We break this goal down into two phases: circumnavigation and landing.\(^{5} \) The reason for this is two-fold. First, we allow ourselves the ability to define two sub-problems, each with different constraints. Second, we gain flexibility by being able to change control objectives, and controller parameters between problems. The main focus of this work will be on the first phase, circumnavigation, because this is where the issue of non-convexity presents itself most prominently in our set up.

One of our main control objectives is to achieve the mission goal while using the least amount of fuel. We can therefore re-state our problem in a way more amenable to mathematical representation: How can we design control inputs to guarantee a safe landing of the spacecraft sufficiently near the target, while minimizing the amount of fuel consumed?\(^a \) The overall problem can be broken down into two sub-problems as follows.

**Problem 1 (Circumnavigation).** Given the circumnavigation target state \( \mathbf{x}_T \), spacecraft state at time \( k \), \( \mathbf{x}_k \), and prediction horizon \( N \), solve

\[
\min_{\mathbf{u}_k, \ldots, \mathbf{u}_{k+N-1}} J = \frac{1}{2} (\mathbf{x}_T - \mathbf{x}_N)^T Q_f (\mathbf{x}_T - \mathbf{x}_N) + \sum_{i=0}^{N-1} (\mathbf{x}_T - \mathbf{x}_{k+i})^T Q (\mathbf{x}_T - \mathbf{x}_{k+i}) + \mathbf{u}_{k+i}^T R \mathbf{u}_{k+i}
\]

subject to:

\[
\mathbf{x}_{k+i} = A \mathbf{x}_{k+i-1} + B \mathbf{u}_{k+i-1} + \mathbf{f}_g(\mathbf{x}_{k+i-1}), \quad i = 1, \ldots, N
\]

(7)

\[
\|\mathbf{u}_{k+i}\| \leq u_{max}, \quad i = 0, \ldots, N - 1
\]

(8)

\[
\mathbf{x}_{k+i} \in \mathbb{R}^6 \setminus B, \quad i = 1, \ldots, N
\]

(9)

\(^a \)By sufficiently near, we mean that a typical mission plan would provide a landing ellipse centered at a point, rather than that singular point. Sufficiently near therefore means within the landing ellipse.
where $B$ defines the shape of the small body, $u_{\text{max}}$ is the maximum allowable control input, and $Q, R, Q_f \in \mathbb{S}_{++}^n$ are positive definite weighting matrices.

**Problem 2 (Landing).** Given the landing target state $x_L$, spacecraft state at time $k$, $x_k$, and prediction horizon $N_L$, solve

$$
\min_{u_k, \ldots, u_{k+N_L}} J = \frac{1}{2} (x_L - x_{N_L})^T Q_f (x_L - x_{N_L}) + \sum_{i=0}^{N_L-1} (x_L - x_{k+i})^T Q (x_L - x_{k+i}) + u_{k+i}^T R u_{k+i}
$$

subject to:

1. $x_{k+i} = A x_{k+i-1} + B u_{k+i-1} + f(x_{k+i-1}), \quad i = 1, \ldots, N_L$ (11)
2. $\|H(x_{k+i} - x_L)\| \cos \theta - (x_{k+i} - x_L)^T H^n L \leq 0, \quad i = 1, \ldots, N_L$ (13)

where $H = [I_3 \ 0_3], n_L$ is the surface normal at the landing site, $\theta$ is the glide-slope angle, $u_{\text{max}}$ is the maximum allowable control input, and $Q, R, Q_f \in \mathbb{S}_{++}^n$ are positive definite weighting matrices. Note that these quantities need not be the same as in Problem 1, but are written the same for clarity.

Switching between phases is accomplished once the spacecraft has reached a predefined distance from the circumnavigation target, $x_T$. Finally, the entire process is terminated once the spacecraft reaches a predefined distance from the landing target, $x_L$. See Section V for more details.

**Remark III.1.** Problem 1 is a non-convex problem due to (9). Convexifying this constraint is a main focus of this work. Problem 2 is a second order cone problem as stated, which is a convex problem and can be readily solved by available solvers.

### IV. Convexification

The main issue with Problem 1 is that the last constraint is highly dependent on the shape of the body, which can be irregular. As a result the constraint is usually nonconvex – making the problem difficult to solve to global optimality in finite time.

To convexify the problem, we must somehow alter the constraints in such a way that they become convex and solutions of the new problem represent feasible solutions of the original. Some work has already been done in this area. A rotating hyperplane method is used in Refs. 4 and 5 whereby an affine approximation to a concave constraint is rotated around a circumscribing ellipsoid with constant velocity. In Ref. 7 sequential quadratic programming (SQP) is applied, wherein the nonconvex constraint is linearized and a solution is obtained iteratively, starting from a given trajectory. Each of these is using the same basic idea; an affine representation of a concave constraint is the closest we can get to a convex representation. The rotating hyperplane method is more heuristic, and mathematical proofs of convergence have not been reported to date.

Following the work in Ref. 5, we’ll define a circumscribing ellipsoid that encompasses the object to be avoided; in our case, the asteroid. Let $W \in \mathbb{S}_{++}^3$ define this ellipsoid. The constraint (9) can then be parameterized as

$$
x_{k+i}^T H^T W H x_{k+i} > 1, \quad i = 1, \ldots, N,
$$

where $H = [I_3 \ 0_3] \in \mathbb{R}^{3 \times 6}$. Swapping the equation above with (9) in Problem 1 yields a new problem with a concave constraint. It is the effort of this work to give a new form of the hyperplane method that allows us to convexify (14) and arrive at a convex approximation to Problem 1.

#### A. Optimal Hyperplane Method

As stated above, we will now discuss the reformulation of Problem 1 into a convex problem. In previous works, the affine hyperplane constraint is simply rotated around the asteroid at a constant rate. The rate at which the hyperplane is rotated becomes a parameter of the problem, and it is not obvious what it should be.

Our first objective is to systematically define a separating hyperplane using only the current location of the spacecraft and knowledge of the circumscribing ellipsoid. Let the circumscribing ellipsoid be given by the set $W$, mentioned above. By design $W$ is a convex set, and hence for each $\rho \in \partial W$ there is a supporting hyperplane to $W$ at the point $\rho$. For each spacecraft position vector, $r$, there is a set of supporting...
hyperplanes to $W$ that form separating hyperplanes between the point $r$ and the set $W$. By projecting the current spacecraft position vector onto the set $W$, we obtain one such separating hyperplane. The plane defined by the normal vector pointing from the projected point towards the spacecraft is this hyperplane. See Figure 1 for an illustration.

To obtain the separating hyperplane, we first solve the following convex optimization problem.

$$\begin{align*}
\min_{\rho} \ |r - \rho|^2 \\
\text{subject to} \quad \rho^T W \rho \leq 1
\end{align*}$$

(15)

Proposition 1. The solution to the optimization problem (15) can be written as

$$\rho^* = (I + \lambda W)^{-1} r,$$

(16)

where $r$ is the current spacecraft position vector, $W$ describes the ellipsoid, and $\lambda$ is the solution to

$$r^T (I + \lambda W)^{-1} W (I + \lambda W)^{-1} r = 1.$$  

(17)

Proof. The optimization process essentially finds the closest point on the ellipsoid $W$ to the spacecraft, which is $\rho^*$. The Weierstrass Theorem guarantees the existence of $\rho^*$, but this is also clear from context.\textsuperscript{33}

We write the Lagrangian for the problem (15) as

$$L(\rho, \lambda) = (r - \rho)^T (r - \rho) + \lambda (\rho^T W \rho - 1).$$

(18)

The first order necessary conditions for optimality are then

$$\frac{\partial L}{\partial \rho} = -2r + 2\rho^* + 2\lambda W \rho^* = 0,$$

(19)

$$\frac{\partial L}{\partial \lambda} = \rho^{*T} W \rho^* - 1 = 0,$$

(20)

which yields,

$$\rho^* = (I + \lambda W)^{-1} r,$$

(21)

$$\rho^{*T} W \rho^* = 1,$$

(22)

since the matrix $I + \lambda W$ is diagonal and positive definite. The first equation establishes (16). Putting (16) into (22) gives

$$r^T (I + \lambda W)^{-1} W (I + \lambda W)^{-1} r = 1.$$  

(23)

which establishes (17).
A line search was performed in the direction of \( \rho^* \) until we arrive at exactly \( \rho^T W \rho^* = 1 \).\(^{b}\)

Next, the outward facing optimal hyperplane normal can be computed according to

\[
\eta = \frac{r - \rho^*}{\|r - \rho^*\|}.
\] (24)

The constraint ensures that the spacecraft and asteroid always lie on opposite sides of the hyperplane defined by \( \eta \). We thus have a linear constraint on the position of the spacecraft, which can be used to re-formulate Problem 1 in the following way.

**Problem 3.** Given the circumnavigation target state \( x_T \), spacecraft state at time \( k \), \( x_k \), and prediction horizon \( N \), solve

\[
\begin{align*}
\min_{u_k,...,u_{k+N-1}} & \quad J = \frac{1}{2} (x_T - x_N)^T Q_f (x_T - x_N) + \sum_{i=0}^{N-1} (x_T - x_{k+i})^T Q (x_T - x_{k+i}) + u_{k+i}^T R u_{k+i} \\
\text{subject to:} & \quad x_{k+i} = A x_{k+i-1} + B u_{k+i-1} + f_\delta(x_{k+i-1}), \quad i = 1, \ldots, N \\
& \quad \|u_{k+i}\|_\infty \leq u_{\text{max}}, \quad i = 0, \ldots, N - 1 \\
& \quad \eta_k^T (H x_{k+i} - \rho^*_k) \geq 0, \quad i = 1, \ldots, N
\end{align*}
\] (25)

where \( W \) defines the ellipsoidal shape completely encompassing the small body, \( u_{\text{max}} \) is the maximum allowable control input, and \( Q, R, Q_f \in \mathbb{S}_++ \) are positive definite weighting matrices.

Problem 3 is now a convex problem and can be solved using commercial solvers such as CVX.\(^{34,35}\)

**Remark IV.1.** The optimal separating hyperplane defined by \( \eta_k \) is a separating hyperplane for all discrete times \( k + i, \; i = 1, \ldots, N \), where \( N \) is the prediction horizon. The constraint is not updated while solving the Problem 3, but instead only prior to re-solve at time \( k + 1 \).

**B. Projection Method**

One of the main issues with any hyperplane method – rotating or optimal – is that for a large portion of spacecraft states, the target point, \( x_T = [r_T^T, \; v_T^T]^T \), used in the MPC problem is infeasible. This is illustrated in Figure 1, where the target lies in the infeasible region defined by the hyperplane. For this reason, when we use the true final target as the set-point in Problem 3, we cannot guarantee a priori that it will be reached.

This issue serves as the motivation for the current section. We want to redefine the set-points used in the Problem 3 so that we can guarantee stability for a new problem. Then, we argue that these new set-points must converge to the true target point, \( x_T \), under the dynamics considered. What results is a sequence of MPC problems, each being feasible and stable, which ultimately steers the spacecraft to the desired target state.

By examining solutions of Problem 3, we observed that each solution of the MPC problem would drive the spacecraft towards the projection of the target point onto the current separating hyperplane. This makes sense when one considers the static hyperplane over a single horizon and the quadratic cost function. In order to minimize the cost, one needs to shrink the vector \( (x_T - x) \), while reducing the control to zero as \( k \to k + N \). The feasible point that is closest to the desired target state would be exactly its projection onto the separating hyperplane (with zero velocities in both cases). This formed the idea for how to choose the new set-points.

At each re-solve time, we must first determine whether or not the true target point is feasible or not. This is a simple matter of checking the following geometric condition. At time \( k \), if

\[
(r - \rho^*, \; r_T - \rho^*) \geq 0,
\] (29)

where \( x = [r^T \; v^T]^T \), then the target location lies in the feasible region for the hyperplane computed using (24). If this condition is not satisfied, then we must alter our MPC problem so that the set-point used in the cost function is feasible.

Let us denote by \( r_P \) the projection of \( r_T \) onto the separating hyperplane defined by \( \eta \). This is the blue dot in Figure 1. A hyperplane is an affine set - a linear subspace translated by some vector. This translation vector is exactly \( \rho^* \), and therefore we can write the affine set as

\[
A = \{z \in \mathbb{R}^3 | z = m + \rho^*, \; m \in M\},
\] (30)

\(^{b}\)Within a specified tolerance, see Figure 4.
where \( M \) is some two-dimensional linear subspace of \( \mathbb{R}^3 \). We will obtain an exact expression for \( r_P \) using the projection theorem, since \( \mathbb{R}^3 \) is a Hilbert space.\(^{36}\) Since \( r_P \in A \) is the projection of \( r_T \) onto \( A \), we have

\[
\begin{cases}
  r_P \in M, & \text{feasibility} \\
  r_T - r_P \perp M, & \text{optimality}
\end{cases}
\]

In order to apply the projection theorem, we need basis vectors that characterize the subspace \( M \). By construction, \( \eta \) is a unit vector orthogonal to \( M \). Further, since \( M \) is two-dimensional we require two orthogonal vectors to fully describe its elements. To find such vectors, we turn to the Gram-Schmidt procedure. By choosing two random initial vectors, and taking \( e \) orthogonal vectors to fully describe its elements. To find such vectors, we turn to the Gram-Schmidt algorithm to obtain

\[
\{ \eta, v_1, v_2 \} \rightarrow \{ \eta, e_1, e_2 \},
\]

where \( \eta \) remains unchanged since it is already a unit vector. Now, \( e_1 \) and \( e_2 \) form a basis for the subspace \( M \). The projection theorem then gives us the point \( r_P \) as

\[
r_P = \rho^* + \langle r_T, e_2 \rangle e_2 + \langle r_T, e_3 \rangle e_3.
\]

Under the dynamics given by (3), we have the following proposition.

**Proposition 2.** If \( x_T = [r_T^T, v_T^T]^T \) is infeasible at time \( k + 1 \), then

\[
\| r_T - r_{k+1} \| < \| r_T - r_k \|,
\]

where \( r_k \) (\( r_{k+1} \)) is the spacecraft position vector at time \( k \) (\( k+1 \)), and \( r_{k+1} \) is related to \( r_k \) through the discretized dynamics given by (3). That is, the spacecraft moves closer to the target position under successive control actions obtained by solving Problem 4.

**Proof.** The discrete relative position dynamics in the asteroid fixed frame can be written as

\[
r_{k+1} = r_k + \Delta t v_k,
\]

where \( \Delta t \) is the sampling time.

Denote the angle between \( r_T \) and \( r_{P,k} \) by \( \theta_k \), and the angle between \( r_T \) and \( r_{P,k+1} \) by \( \theta_{k+1} \). Since \( \theta_{k+1} \) represents an infeasible location, so must \( \theta_k \), and hence \( \theta_k, \theta_{k+1} \in [\pi, 2\pi] \). Since the control input is generated by solving Problem 4, we know that the optimal cost is monotonically decreasing by applying the results of\(^6\) Since our cost is a quadratic function of state and control, \( \theta_{k+1} \leq \theta_k \) under the control input \( u_k \). This means that \( r_{P,k+1} \) lies closer to \( r_T \) than \( r_{P,k} \) does, and

\[
- \cos \theta_k \leq - \cos \theta_{k+1}.
\]

Using the law of cosines, we then have

\[
\| r_T - r_{k+1} \|^2 = \| r_T \|^2 + \| r_{k+1} \|^2 - 2 \| r_T \| \| r_{k+1} \| \cos \theta_{k+1}
\]

\[
= \| r_T \|^2 + \| r_k + \Delta t v_k \|^2 - 2 \| r_T \| \| r_k + \Delta t v_k \| \cos \theta_{k+1}
\]

\[
\leq \| r_T \|^2 + (\| r_k \| + \| \Delta t v_k \|)^2 - 2 \| r_T \| (\| r_k \| + \| \Delta t v_k \|) \cos \theta_{k+1}
\]

\[
\leq \| r_T \|^2 + \| r_k \|^2 + 2 \| r_k \| \| \Delta t v_k \| + \| \Delta t v_k \|^2 - 2 \| r_T \| \| r_k \| \cos \theta_{k+1} - 2 \| x_T \| \| \Delta t v_k \| \cos \theta_{k+1}
\]

\[
\leq (\| r_T \|^2 + \| r_k \|^2 - 2 \| r_T \| \| r_k \| \cos \theta_{k+1}) + 2 \| \Delta t v_k \| \left( \| r_k \| + \frac{1}{2} \| \Delta t v_k \| - \| r_T \| \cos \theta_{k+1} \right)
\]

\[
\leq (\| r_T \|^2 + \| r_k \|^2 - 2 \| r_T \| \| r_k \| \cos \theta_{k+1}) + 2 \| \Delta t v_k \| \left( \| r_k \| + \frac{1}{2} \| \Delta t v_k \| - \| r_T \| \cos \theta_{k+1} \right)
\]

\[
\leq \| r_T - r_k \|^2
\]

where the last inequality follows from the fact that infeasibility of \( x_T \) at time \( k + 1 \) gives us \( \theta_{k+1} > 90^\circ \), implying that \( \cos \theta_{k+1} < 0 \) and each term in the second bracket is strictly greater than zero. The second to last inequality uses (35). We have therefore established that by using solutions of Problem 4, the distance from both the spacecraft and the new set-points to the desired target location are monotonically decreasing sequences when the dynamics (3) are assumed valid.
The above proposition is saying the following. By redefining the target point to be the projection of \( r_T \) onto the optimal hyperplane defined by \( \eta_k \), we have obtained a new target point \( x_{P,k} \). Note that \( x_{P,k} \) is obtained from \( r_{P,k} \) by appending the zero velocity vector. We have shown above that under the discretized dynamics, the angle between the vectors \( r_T \) and \( r_{P,k} \) is decreasing, which implies that eventually, \( r_T \) becomes a feasible point (hence so does \( x_T \)), and a collision-free straight line path exists between the spacecraft position and the target. Further, we established that the distance between the spacecraft and the target point is strictly decreasing - provided the spacecraft moves towards the auxiliary target points, \( r_{P,k} \).

We are now ready to state our second convexified version of Problem 1. Using the projection theorem method described above, the new problem is as follows.

**Problem 4.** Given the target state \( x_T \) and current spacecraft state \( x_k \), solve

\[
\min_{u_k, \ldots, u_{k+N-1}} J = \frac{1}{2} (x_{P,k} - x_N)^T Q_f (x_{P,k} - x_N) + \sum_{i=0}^{N-1} (x_{P,k} - x_{k+i})^T Q (x_{P,k} - x_{k+i}) + u_{k+i}^T Ru_k \tag{36}
\]

subject to:

\[
x_{k+i} = Ax_{k+i-1} + Bu_{k+i-1} + f_g(x_{k+i-1}), \quad i = 1, \ldots, N \tag{37}
\]

\[
\|u_{k+i}\|_\infty \leq u_{max}, \quad i = 0, \ldots, N - 1 \tag{38}
\]

\[
\eta_k^T (H x_{k+i} - \rho_k^*) \geq 0, \quad i = 1, \ldots, N \tag{39}
\]

where \( \eta_k \) is the solution to (24), \( u_{max} \) is the maximum allowable control input, and \( Q, R, Q_f \in \mathbb{S}^n_{++} \) are positive definite weighting matrices.

**Remark IV.2.** The last constraint in Problem 4 may no longer be strictly necessary. The reason is that \( r_{P,k} \) is guaranteed to lie some distance away from the set \( W \) that encompasses the asteroid, unless the initial spacecraft position vector lies anti-parallel to the target, \( r_T \). Small perturbations off this line, nearly guaranteed by gravitational forces, easily avoid this case. This phenomenon is very similar to local minima arising from in use of Koditscheck-Rimon navigation functions when the initial condition is aligned with the direction of the eigenvector corresponding to the minimum eigenvalue of the objective function.\(^{37}\)

**C. Regularization of the Inputs**

A weighting term can be added to the cost function \( J \) in both Problem 4 and 2 in order to regulate the control inputs. This reduces any jitter in the inputs that would not be realizable in practice. We add a term in the cost function,

\[
\lambda \|u_k - u_{k-1}^*\|, \tag{40}
\]

that penalizes deviations in the optimal control input from one solution to another. Note that \( u_{k-1}^* \) is the optimal solution applied to the system at the previous time step. The quantity \( \lambda \geq 0 \) is used as a tuning parameter that weights how much or how little we would like the inputs to deviate from one time instance to another.

**V. Results**

We now present the numerical results of the convexification procedure described above. To illustrate all aspects of the work, we solve Problems 4 and 2 in sequence, starting from a challenging initial condition. The chosen initial condition lies on the opposite side of the landing site, so that the spacecraft is forced to circumnavigate to the other side of the body.

We allow the spacecraft to switch phases once it satisfies \( |r_k - r_T| < \varepsilon_C \) for some tolerance \( \varepsilon_C \) specific to the circumnavigation phase. Once this is achieved, the final state from the circumnavigation phase is used as the initial condition for the landing phase and we continue until \( |r_k - r_L| < \varepsilon_L \). The simulation parameters are summarized in Table 1.

The spacecraft trajectory is presented in Figure 2. Two views are given to show how the trajectory avoids the asteroid and switches to the landing phase. The spacecraft successfully avoids the asteroid and reaches the surface of the asteroid with a low velocity 0.81 meters from the landing site.

Figure 3 gives the time history of position, velocity and control input of the spacecraft. The control input can be seen to remain within the control bounds at all times. Figure 4 shows the level of constraint satisfaction. For the circumnavigation phase, we measure how close the optimal hyperplane and projection
methods are working by checking the value of $\rho^T W \rho^*$. This value should be close to 1 as per Proposition 2. The tolerance of the line search was set to 0.01, which means that we should expect

$$0.99 \leq \rho^T W \rho^* \leq 1.01,$$

which is supported by Figure 4(a). Also in Figure 4(a), we see the glide slope constraint imposed during the landing phase. At all times, this is below the threshold of 30°, as expected. In Figure 4(b), we see evidence that the auxiliary set points, $x_{P,k}$ do indeed converge to the true target, $x_T$. We can see that the normed difference strictly decreases, as expected from Proposition 2.

Finally, Table 2 presents the final position and velocity results of each phase. The spacecraft landed close to the landing target with a low velocity. Based on the results we conclude that the convexification procedure is working properly and producing feasible trajectories to the original problem (Problem 1).

![Figure 2](image-url)

(a) $Y - Z$ axis view.  
(b) $X - Y$ axis view.

Figure 2. Trajectory results from the optimal hyperplane and projection theorem procedures described above.

<table>
<thead>
<tr>
<th>Circumnavigation</th>
<th>Landing</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_T$</td>
<td>$x_L$</td>
</tr>
<tr>
<td>$Q, P$</td>
<td>$Q, P$</td>
</tr>
<tr>
<td>$R$</td>
<td>$R$</td>
</tr>
<tr>
<td>$N$</td>
<td>$N$</td>
</tr>
<tr>
<td>$(u_{min}, u_{max})$</td>
<td>$(u_{min}, u_{max})$</td>
</tr>
<tr>
<td>$\varepsilon_C$</td>
<td>$\varepsilon_L$</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>$\lambda$</td>
</tr>
<tr>
<td>$x_0$</td>
<td>$\theta$</td>
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</table>

<table>
<thead>
<tr>
<th>Circumnavigation</th>
<th>Landing</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta r_C$</td>
<td>$\Delta r_L$</td>
</tr>
<tr>
<td>$\Delta v_C$</td>
<td>$\Delta v_L$</td>
</tr>
</tbody>
</table>

Table 2. Deviations from desired target states for both phases.
(a) Position and velocity time history, along with the target landing site coordinates.  
(b) The control input history with control bounds.

Figure 3. Simulation results using the parameters in Table 1.

(a) Constraint satisfaction in both phases. During the circumnavigation phase we want (41) to hold, and during the landing phase we require the glide slope angle to be less than 30°.  
(b) The norm difference between $x_P$ and $x_T$ during the circumnavigation phase, confirming the results obtained in Proposition 2.

Figure 4. Constraint satisfaction and projection method showing decreasing normed distance.

VI. Conclusion

We have presented a new method for convexifying the small body soft-landing problem. The method builds on the rotating hyperplane method that has been used in the past, but computes the hyperplanes by solving a separate convex problem. A semi-analytic solution coupled with a simple line search algorithm allows us to define optimal separating hyperplanes at each re-solve time. We then use a projection theorem argument to guarantee feasible set points for the MPC problem; a novel feature. We further established that the distance from both the new set points and spacecraft position vector to the target location strictly decreases. These theoretical findings were corroborated with successful numerical simulations. The results are the first that use the hyperplane method in this way, and gives the first non-heuristic argument of convergence to the desired target state.

There are several areas for future work. Foremost, exploring the use of Robust MPC would serve to strengthen the results given here. In Robust MPC, the state is split into a nominal state and a perturbed state, with the control law derivations given here serving as the nominal control. A feedback policy that is robust to disturbances is then derived for the perturbed state, and the two combine to form the overall control law. The work in Ref. 6 provides an MPC and LMI based approach to solving this problem for the landing phase only. An extension to the circumnavigation phase would be a natural next step. It would also do well to consider rigid body dynamics. Dual quaternions could be used to parameterize both rotational and translational motion and provide the most accurate representation of the true motion.8
Acknowledgments

The author would like to thank Krish Kaycee from Planetary Resources and UnsiK Lee for many conversations about asteroid landing, MPC, and spacecraft GN&C.

References


