Multiple Time-Scales in Network-of-Networks

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Abstract—**This paper examines a multiple time-scale system where each layer of time-scale dynamics corresponds to a network. Coupling between layers induces a network-of-networks dynamic system. Assuming a hierarchical interaction structure between network time-scale layers with sufficient time-scale differences, layers can be studied under a separation principle. We describe stability of the network-of-networks through a composite Lyapunov function and provides a bilinear matrix inequality condition to guarantee asymptotic stability. Techniques are proposed to adapt this inequality into a linear matrix inequality condition making it computationally efficient to solve as a convex optimization problem. We examine estimates of the dynamics' domain of attraction and provide conservative bounds on the time-scale separation required to guarantee stability. A class of network-of-networks dynamics with modified consensus dynamics and state-dependent network structure is explored. Graph-based guarantees are provided that certify the stability of the system with the worst case rate of convergence dictating conservative bounds on the time-scale separation. An illustrative network-of-networks example is presented.**

Index Terms—**Network-of-networks; Time-Scale Separation; Networked Dynamic Systems; State-Dependent Networks**

I. Introduction

Networks are an integral part of our technological world as well as core to our understanding of complex phenomena arising from multifaceted interactions among dynamic systems [1], [2]. In recent years, this network-centric point of view of complex systems has been blended quite extensively with that of dynamical systems and control. This new area of control theory has led to not only new insights on the richness of networked behavior, but a wide range of open problems at the intersection of networks and graph theory, dynamical systems, and control.

In this direction, two notable areas of significant interest in control theory include that of multi-layered systems and state-dependent networks. Multi-layered systems arise in large scale networks due to modularity and hierarchy - architectural principles that naturally arise in physical, biological, and social settings, as well as providing a framework for designing large scale engineering systems [3], [4], [5]. State-dependencies, on the other hand, arise when interactions among the dynamic nodes in the network have dynamics of their own, responding in turn to the state of the nodes that they are incident on. In this case, it is conceivable that node interactions that influence the edge dynamics, that in turn feedback to the node dynamics, can lead to complex nonlinear behavior, including chaotic dynamics, bifurcation, clustering, among others [6], [7], [8], [9], [10]. Layering state-dependent networks does not alleviate the inherent complexity of analyzing these systems.

This paper considers a natural twist on the analysis and control of multi-layered systems through a more systematic focus on their time-scale separation. The basic thesis underlying our presentation is that reasoning about multi-layered state-dependent networks can be streamlined by exploiting their distinct time-scales. Such a point of view had been adopted in our previous work [11], and appreciated by other researchers in the field. For example, time-scale analysis has been used for model reduction in large-scale power networks by area-aggregation [12], dynamic equivalence [13], and slow coherency [14]. Weak inter-node connections have been examined from a graph-theoretic perspective [15]. In the meantime, singular perturbation methods integral to time-scale separation were leveraged upon in network synthesis problems in [16]. More generally, there is an extensive literature on the use of singular perturbation in control design that provides the needed inspiration for adopting a similar point of view in network control.

The contribution of this work is threefold: 1) A formulation of a network-of-networks system with multiple time-scale and corresponding reduced order models illustrating a separation principle over layers of the network; 2) Quantitative bounds on the time-scale parameters sufficient to provide asymptotic stability of the dynamics using optimization techniques; 3) Analysis of the applicability of the results to an augmented consensus network-of-networks problem with graph theoretic bounds on the sufficient time-scale parameters.

The structure of the paper is as follows. In §II, we present the background material. In §III we examine the general network-of-networks problem with reduced order model and quantitative analysis of time-scales. The augmented consensus multi-time-scale system is examined in §IV. The main results are illustrated with a human-UAV swarm example in §V, and conclusions are presented in §VI.

II. Background

In this section, a brief background is provided on the notation, dynamics and definitions used in this paper. For a column vector $v \in \mathbb{R}^p$, $[v]_i$ denotes its ith element. The identity matrix is denoted as I and e_i is the column vector with all zero entries except $[e_i]_i = 1$. The column vector of all ones is denoted as $\mathbf{1}$. The cardinality of a set S is denoted as $|S|$. The eigenvalues of a matrix $M \in \mathbb{R}^{n\times n}$, are represented and ordered as $\lambda_1(M) \le \lambda_2(M) \le \cdots \le \lambda_n(M)$. The diameter

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of a set $D \subset \mathbb{R}^n$ is denoted and defined as $\mathrm{diam}(D) = \max\{\|x-y\|_2 \,|x, y \in D\}$.

The consensus problem is defined on an undirected, weighted graph $\mathcal{G} = (V, E, W)$ that consists of a node set V with cardinality n, an edge set E with cardinality m, and a positive weight set W with cardinality m and associated vector of weights w under some ordering [2]. The incidence matrix $D(\mathcal{G})$ is a $n \times m$ matrix with columns $e_i - e_j$ for each edge $\{i, j\} \in E$, by construction $\mathbf{1}^T D(\mathcal{G}) = 0$. For the consensus dynamics, $[x(t)]_i \in \mathbb{R}$ represents the state of agent $i \in V$ at time t, and each agent's states evolve directly relative to the state measurements of neighboring agents in the graph $\mathcal{G}$. The dynamics are compactly represented over all agents in the graph as

$$\dot{x}(t) = -L(\mathcal{G})x(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $L(\mathcal{G}) = D(\mathcal{G})WD(\mathcal{G})^T$ is the Laplacian matrix whose eigenvalues $0 = \lambda_1(L(\mathcal{G})) \leq \cdots \leq \lambda_n(L(\mathcal{G}))$ are associated with the normalized eigenvectors $\frac{1}{n}\mathbf{1} = v_1, v_2, \ldots, v_n$. For brevity, $\lambda_i(L(\mathcal{G}))$ is denoted as $\lambda_i(\mathcal{G})$. An attraction of these dynamics is that all agents converge to the consensus subspace $\mathcal{X}_c = \{x \in \mathbb{R}^n \mid [x(t)]_1 = \cdots = [x(t)]_n\}$ when $\mathcal{G}$ is a connected graph. Special graphs are the path graph with $E = \{\{i, i+1\} | i = 1, \ldots, n-1\}$; the star graph with $E = \{\{1, i\} | i = 2, \ldots, n\}$; and the complete graph with $E = \{\{i, j\} | i = 1, \ldots, n-1, j = i+1, \ldots, n\}$.

For a small, positive perturbation parameter ε, a system

$$\begin{aligned} \dot{x} &= f(x, z; \varepsilon)\,;\; x(t=0) = x(0) \\ \varepsilon \dot{z} &= g(x, z; \varepsilon);\; z(t=0) = z(0) \end{aligned} \tag{2}$$

(with x not necessarily a set of agent states) is said to be in *standard singularly perturbed form* if the vector fields are bounded and continuously differentiable in their arguments, and if each root $z = h(x)$ of the algebraic equation $0 = g(x, z; 0)$, found by setting $\varepsilon = 0$ in (2), is isolated [17]. The states $x \in D_x \subset \mathbb{R}^{n_s}$ are then called the *slow states* and $z \in D_z \subset \mathbb{R}^{n_f}$ the *fast states*.

Of particular interest are the intuitive notions of the *Reduced Slow System* which describes the dynamics of the slow states $x^{(0)}$ of (2) as if the fast states are fixed at an isolated equilibrium $h(x)$, and the *Reduced Fast System* which describes the dynamics of the fast states of (2) over a stretched time-scale $\tau = t/\varepsilon$ for which the $x^{(0)}$ are assumed constant. The results in this paper make use of the Tikhonov theorem in the singular perturbation literature, which states the conditions over which these reduced order models are valid (see [17] for details).

III. Multiple Time Scale Networked Systems

An attraction of multiple time-scale systems is the potential to decompose the analysis of the dynamics into its individual time-scale layers and their respective couplings between layers. When each time-scale represents a dynamic network, then the separation corresponds to an inter-network analysis and intra-network analysis. In general, coupling can occur between arbitrary time-scales of a system. Motivated by a hierarchical network structure in this paper, we consider a specific form whereby a network time-scale layer is only directly coupled with the closest faster and closest slower time-scale layers. This structure is summarized in Figure 1.

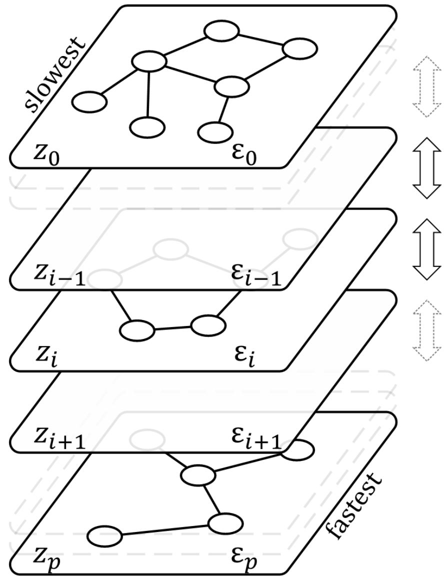


Figure 1. Conceptual image of a multiple time-scale networked system.

We consider a $(p+1)$-layer network over $(p+1)$ time-scales. Here, network layer i has n_i nodes with node state dynamics $z_i(t) \in \mathbb{R}^{n_i}$ and is associated with the small parameter ε_i which captures its time scale properties. For convenience we assume that $\varepsilon_0 > \varepsilon_1 > \cdots > \varepsilon_p > 0$ and so layer 0 is the slowest network layer and p is the fastest. As mentioned previously for $1 < i < p-1$ the state dynamics $z_i(t)$ are directly coupled to the state dynamics of networks $i-1$ and $i+1$ through the nonlinear function $g_i(z_{i-1}(t), z_i(t), z_{i+1}(t))$. For notational compactness, the slowest state dynamics $z_0(t)$ is assumed to be coupled to $z_1(t)$ and dummy state $z_{-1}(t) = 0$ for all t, and similarly for the fastest state dynamics $z_p(t)$ with dummy state $z_{p+1}(t) = 0$ for all t. The combined dynamics are subsequently summarized in the following singularly perturbed model (henceforth for brevity we shall omit t):

$$\begin{aligned} \varepsilon_0 \dot{z}_0 &= g_0(z_{-1}, z_0, z_1) \\ \varepsilon_1 \dot{z}_1 &= g_1(z_0, z_1, z_2) \\ &\vdots \\ \varepsilon_p \dot{z}_p &= g_p(z_{p-1}, z_p, z_{p+1}) \end{aligned} \tag{3}$$

where $z_{-1} = z_{p+1} = 0$. Now consider the following assumptions on the nonlinear functions $g_i(z_{i-1}, z_i, z_{i+1})$.

Assumption 3.1. *Each function $g_i(z_{i-1}, z_i, z_{i+1})$ is continuously differentiable with respect to z_{i-1}, z_i, and z_{i+1} over its respective domains $D_{i-1} \subset \mathbb{R}^{n_{i-1}}$, $D_i \subset \mathbb{R}^{n_i}$ and $D_{i+1} \subset \mathbb{R}^{n_{i+1}}$, and has a unique, isolated zero $(z_{i-1}, z_i, z_{i+1}) = (0, 0, 0)$.*

Assumption 3.2. *There exists a positive-definite Lyapunov function $V_i(z_{i-1}, z_i)$, with respect to z_i, for all $z_{i-1} \in D_{i-1}$ and $z_i \in D_i$ such that*

$$\frac{V_i(z_{i-1}, z_i)}{\partial z_i} g_i(z_{i-1}, z_i, 0) \leq -\alpha_i \Phi_i^2(z_i),$$

with $\alpha_i > 0$ and $\Phi_i(z)$ a continuous scalar function with $\Phi_i(0) = 0$.

These assumptions are common for singular perturbation systems when $g_i(\cdot)$ is independent of t and ε_i. Assumption 3.1 describes a dynamics with equilibrium at the origin and Assumption 3.2 indicates that the origin $z_i = 0$ is asymptotically stable uniformly for all values of z_{i-1}. We note that Assumption 3.2 describes properties of $g_i(\cdot)$ for $z_{i+1} = 0$. The motivation here is that for a sufficiently fast network layer $i + 1$ compared to layer i, the dynamics in layer i will preserve z_{i+1} as already converged to the origin.

In Section IV, we apply additional structure to dynamics (3) to encode the network structure into the problem specializing dynamics $g_i(z_{i-1}, z_i, z_{i+1})$. In the remainder of this section, we examine the *general* dynamics (3) using standard techniques from geometric singular perturbation theory.

A. Separation Principle

The singular perturbation form (3) indicates that the system exhibits multiple time-scale behavior as $\varepsilon_i/\varepsilon_{i-1} \to 0$. To aid the evaluation of this question, a new stretched time variable $\tau_i = t/\varepsilon_i$ is introduced. In this time-scale, the variables z_j for $j < i$ are slowly varying. Applying Assumption 3.1 and 3.2, a ith reduced system dynamics can be formed over a new stretched time variable $\tau_i = t/\varepsilon_i$.

Definition 3.3. The ith Reduced System of (3) is

$$\begin{aligned} \dot{z}_0^{(i)} &= 0 \\ &\vdots \\ \dot{z}_{i-1}^{(i)} &= 0 \\ \dot{z}_i^{(i)} &= g_i(z_{i-1}^{(i)}, z_i^{(i)}, 0) \\ z_{i+1}^{(i)} &= 0 \\ &\vdots \\ z_p^{(i)} &= 0, \end{aligned} \tag{4}$$

where $z_0^{(i)}(0), z_1^{(i)}(0), \ldots, z_p^{(i)}(0) = z_0(0), z_1(0), \ldots, z_p(0)$.

The ith Reduced System describes the dynamics of the states when the $i+1, \ldots, p$ fast states have reached equilibrium.

For a given starting initial state of the model (3), a natural question is how well does $z_0^{(i)}, z_1^{(i)}, \ldots, z_p^{(i)}$ estimate $z_0, z_1, \ldots, z_p$. This is addressed in the following theorem.

Theorem 3.4. *Under the dynamics (3) and with Assumptions 3.1 and 3.2, there exists an $\varepsilon_i^* > 0$, such that for $0 \leq \varepsilon_i \leq \varepsilon_i^*$ then for the ith reduced dynamics (4) then $z_i^{(i)} - z_i = O(\epsilon_i)$ holds uniformly for some interval $t \in [t_1, t_2]$ where $0 < t_1 < t_2$.*

Proof: The result follows from Tikhonov's theorem [17], and the formulation of the problem in standard form. ■

Theorem 3.4 states the conditions under which the reduced models (4) are good approximations of the system.

B. Bounds of ε_i

While Theorem 3.4 provides a certificate of the existence of an ε_i below which the given approximations are valid, a quantitative result is required for implementation purposes. Specifically, there are practical limitations on how quickly updates within each layer can be performed.

To this end a quantitative bound on ε_i's can be formed to guarantee convergence of the dynamics (3). The result is summarized in the following theorem.

Theorem 3.5. *Under the dynamics (3) and with Assumptions 3.1 and 3.2, if there exists constants $\beta_i, \gamma_i, \delta_i \geq 0$ for all $i = 0, \ldots, p$ such that*
1. $\frac{\partial V_i}{\partial z_{i-1}} g_{i-1}(z_{i-2}, z_{i-1}, z_i) \leq \beta_i \psi_{i-1}(z_{i-1})\psi_i(z_i) + \gamma_i \psi_i^2(z_i)$
2. $\frac{\partial V_i}{\partial z_i} \left(g_i(z_{i-1}, z_i, z_{i+1}) - g_i(z_{i-1}, z_i, 0)\right) \leq \delta_i \psi_i(z_i)\psi_i(z_{i+1})$
with $\beta_0 = \gamma_0 = 0$ and $\delta_p = 0$, for all $(z_0, z_1, \ldots, z_p) \in D_0 \times D_1 \times \cdots \times D_p$, then the origin is asymptotically stable for all $\varepsilon = [\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_p]^T > 0$, such that $\mathcal{K}(d, \varepsilon)$ in (5) is positive definite for some $d = [d_0, d_1, \ldots, d_p]^T$, where $\sum_{i=0}^p d_i = 1$ and $d_i \in (0, 1)$ for all i.

Proof: Following closely the structure of the derivation of the two time-scale singularly perturbed system in Section 2.3 [17, pp. 35–39], from the Lyapunov functions in Assumption 3.2, consider the composite Lyapunov function

$$\nu(z_0, \ldots, z_p) = \sum_{i=0}^{p} d_i V_i(z_{i-1}, z_i).$$

The derivative of the composite Lyapunov function gives

$$\begin{aligned} \dot{\nu} &= \sum_{i=0}^{p} d_i \frac{\partial V_i}{\partial z_{i-1}} \frac{1}{\varepsilon_{i-1}} g_{i-1}(z_{i-2}, z_{i-1}, z_i) \\ &\quad + d_i \frac{\partial V_i}{\partial z_i} \frac{1}{\varepsilon} g_i(z_{i-1}, z_i, z_{i+1}) \\ &= \sum_{i=0}^{p} d_i \frac{\partial V_i}{\partial z_{i-1}} \frac{1}{\varepsilon_{i-1}} g_{i-1}(z_{i-2}, z_{i-1}, z_i) \\ &\quad + d_i \frac{\partial V_i}{\partial z_i} \frac{1}{\varepsilon_i} g_i(z_{i-1}, z_i, 0) \\ &\quad + d_i \frac{\partial V_i}{\partial z_i} \frac{1}{\varepsilon_i} \left(g_i(z_{i-1}, z_i, z_{i+1}) - g_i(z_{i-1}, z_i, 0)\right). \end{aligned}$$

Applying Assumption 3.2 and bounds (1) and (2) then

$$\begin{aligned} \dot{\nu} &\leq \sum_{i=0}^{p} \frac{d_i}{\varepsilon_{i-1}} \left(\beta_i \psi_{i-1}(z_{i-1})\psi_i(z_i) + \gamma_i \psi_i^2(z_i)\right) \\ &\quad + \frac{d_i}{\varepsilon_i} \alpha_i \psi_i^2(z_i) + \frac{d_i}{\epsilon_i} \delta_i \psi_i(z_i)\psi_i(z_{i+1}) \\ &= \sum_{i=0}^{p} -d_i \left(\frac{\alpha_i}{\varepsilon_i} - \frac{\gamma_i}{\varepsilon_{i-1}}\right) \psi_i^2(z_i) \\ &\quad + \sum_{i=0}^{p} \left(d_i \frac{\beta_i}{\varepsilon_{i-1}} + d_{i-1} \frac{\delta_{i-1}}{\varepsilon_{i-1}}\right) \psi_{i-1}(z_{i-1})\psi_i(z_i). \end{aligned}$$

$$\mathcal{K}(d,\varepsilon)=\begin{bmatrix} d_0\alpha_0 & -\frac{1}{2\varepsilon_0}\left(d_1\beta_1+d_0\delta_0\right) & & 0 \\ -\frac{1}{2\varepsilon_0}\left(d_1\beta_1+d_0\delta_0\right) & d_1\left(\frac{\alpha_1}{\varepsilon_1}-\frac{\gamma_1}{\varepsilon_0}\right) & \ddots & \\ & \ddots & \ddots & -\frac{1}{2\varepsilon_{p-1}}\left(d_p\beta_p+d_{p-1}\delta_{p-1}\right) \\ 0 & & -\frac{1}{2\varepsilon_{p-1}}\left(d_p\beta_p+d_{p-1}\delta_{p-1}\right) & d_p\left(\frac{\alpha_p}{\varepsilon_p}-\frac{\gamma_p}{\varepsilon_{p-1}}\right) \end{bmatrix}, \tag{5}$$

Rearranging, we see that

$$\dot{\nu} \le -\Gamma(\psi_0,\dots\psi_p)^T\mathcal{K}\Gamma(\psi_0,\dots\psi_p),$$

where $\Gamma(\psi_0,\dots\psi_p)=[\psi_0,\psi_1,\dots,\psi_p]^T$. As the inequality is quadratic, if $\mathcal{K}(d,\varepsilon)$ is positive definite then $\dot{v}<0$ and (3) is asymptotically stable. ■

Theorem 3.5 provides a sufficient bound to guarantee the convergence of dynamics (3) given a time-scale variable ε'. In fact, as $\sum_{i=0}^{p} d_i = 1$ and $d_i > 0$ is a linear constraint and linear inequality, respectively, and $\mathcal{K}(d,\varepsilon')\succ 0$ is a linear matrix inequality then the existence of a corresponding d' such that $\mathcal{K}(d',\varepsilon')$ is positive definite can be solved via the following convex optimization problem:

$$\begin{aligned} d' \quad = \quad & \operatorname{argmin} 0 \\ \text{s.t.} \quad & \mathcal{K}(d,\varepsilon')\succ 0 \\ & \sum_{i=0}^{p} d_i = 1 \\ & d_i > 0. \end{aligned} \tag{6}$$

The challenge arises if an optimal ε with respect to some convex cost function, say $q(\varepsilon_0^{-1},\varepsilon_1^{-1},\dots,\varepsilon_p^{-1})$ is required. This is in general difficult as $\mathcal{K}(d,\varepsilon)\succ 0$ is a bilinear matrix inequality in the vectors d and $(\varepsilon_0^{-1},\varepsilon_1^{-1},\dots,\varepsilon_p^{-1})$ [18], [19]. One approach to solving this is an alternating method whereby a feasible $\varepsilon>0$ is fixed, say at ε', and then an optimization is run to find the corresponding d such that $\mathcal{K}(d,\varepsilon')\succ 0$, which is now a linear matrix inequality. This is followed by fixing the newly found $d'=d$ and solving for an optimal ε' under some favorable cost function and the constraint $\mathcal{K}(d',\varepsilon)\succ 0$. The process is then repeated and equivalent to alternating between the optimization problem (6) and the following optimization problem:

$$\begin{aligned} k' \quad = \quad & \operatorname{argmin} q(k) \\ \text{s.t.} \quad & \mathcal{K}(d',k)\succ 0 \\ & k_0 < \rho_0 \\ & k_i < k_{i-1} \text{ for } i \ge 1, \end{aligned} \tag{7}$$

where $k_i = \varepsilon_i^{-1}$ and ρ_0 is an arbitrary positive number, e.g., $\rho_0 = 1$. The disadvantage of this approach is that a feasible ε is required to seed the process and the alternating optimizations may not converge to the optimal ε^*.

An alternative is to optimize the selection of ε over a different criteria. One of practical interest is the selection of d such that there is favorable approximation of the domain of attraction.

C. Domain of Attraction

Theorem 3.5 provides guarantees on the stability of dynamics (3) within some region about the equilibrium point. The size of this region, or domain of attraction, is dictated by the level sets of the Lyapunov function $\nu(z_0,z_1,\dots,z_p)$ within the proof of Theorem 3.5 and specifically whether $\dot{\nu}(z_0,z_1,\dots,z_p)<0$ for $0<\nu(z_0,z_1,\dots,z_p)\le c$ for some constant c. These level sets can be challenging to characterize; an alternative is to use the domain of attraction of the reduced systems (4) to estimate the domain of attraction of the full dynamics (3).

The Lyapunov function $V_i(z_{i-1},z_i)$ in Assumption 3.2 can be used to approximate the domain of attraction of the ith reduced system (4). Specifically, its c_i-level set gives the estimated domain of attraction

$$L_i = \{z_i : V_i(z_{i-1},z_i)\le c_i\}\subset D_i \text{ for } z_{i-1}\in D_{i-1}. \tag{8}$$

Now, as $\nu(z_0,\dots,z_p)=\sum_{i=0}^{p} d_i V_i(z_{i-1},z_i)$ then

$$L=\{(z_0,\dots,z_p):\nu(z_0,\dots,z_p)\le c\}\subset D_0\times D_1\times\cdots\times D_p$$

is included in the domain of attraction of the equilibrium, and using the domain (8) then c can be chosen as

$$c=\min_{\mathbf{1}^T d=1}\{d_0c_0, d_1c_1,\dots,d_pc_p\}$$

which is maximized if d is chosen as

$$d_i = 1/\left(c_i\sum_{j=0}^{p}\frac{1}{c_j}\right) \tag{9}$$

corresponding to $L = L_0\times L_1\times\cdots\times L_p$. A selection of ε can then be performed over some convex cost $q(\varepsilon_0^{-1},\varepsilon_1^{-1},\dots,\varepsilon_p^{-1})$ as described in the previous section using optimization (7).

The optimization provides a potentially isolated solution for ε, meaning that if errors or perturbations occur in ε, the dynamics may no longer be asymptotically stable. The following theorem provides a non-optimization based range of ε for which the solution is stable.

Theorem 3.6. *Under dynamics (3) and the assumptions and variables described in Theorem 3.4, then the origin is asymptotically stable over the domain* $L=L_0\times L_1\times\cdots\times L_p$ *for all* ε *satisfying* $\varepsilon_i=\left(\prod_{j=1}^{i}\zeta_j\right)^{-1}\varepsilon_0$ *where*

$$\varepsilon_0 > \frac{d_1\beta_1+d_0\delta_0}{2d_0\alpha_0},\ \zeta_p > \frac{d_p(\beta_p+2\gamma_p)+d_{p-1}\delta_{p-1}}{2d_p\alpha_p}$$

and for $i=1,\dots,p-1$

$$\zeta_i > \frac{d_i(\beta_i+2\gamma_i)+d_{i-1}\delta_{i-1}}{d_i(2\alpha_i-\delta_i)-d_{i+1}\beta_{i+1}},$$

where L_i and d are defined in (8) and (9), respectively.

Proof: Apply Gershgorin's disk theorem to the rows of $\mathcal{K}(d,\varepsilon)$ in (5) [20]. If $\mathcal{K}(d,\varepsilon)$ is diagonally dominant than $\mathcal{K}(d,\varepsilon)$ is positive definite. The first row of $\mathcal{K}(d,\varepsilon)$ provides the bound

$$\begin{aligned} d_0\alpha_0 &> \frac{1}{2\varepsilon_0}\left(d_1\beta_1 + d_0\delta_0\right) \\ \varepsilon_0 &> \frac{d_1\beta_1 + d_0\delta_0}{2d_0\alpha_0}. \end{aligned}$$

Noting that $\varepsilon_{i+1} = \zeta_i^{-1}\varepsilon_i$ and dividing through by ε_{p-1} then the final row of $\mathcal{K}(d,\varepsilon)$ provides the bound

$$\begin{aligned} d_p\left(\frac{\alpha_p}{\varepsilon_p} - \frac{\gamma_p}{\varepsilon_{p-1}}\right) &> \frac{1}{2}\left(d_p\frac{\beta_p}{\varepsilon_{p-1}} + d_{p-1}\frac{\delta_{p-1}}{\varepsilon_{p-1}}\right) \\ d_p\left(\zeta_p\frac{\alpha_p}{\varepsilon_{p-1}} - \frac{\gamma_p}{\varepsilon_{p-1}}\right) &> \frac{1}{2}\left(d_p\frac{\beta_p}{\varepsilon_{p-1}} + d_{p-1}\frac{\delta_{p-1}}{\varepsilon_{p-1}}\right) \\ d_p\left(\alpha_p\zeta_p - \gamma_p\right) &> \frac{1}{2}\left(d_p\beta_p + d_{p-1}\delta_{p-1}\right) \\ \zeta_p &> \frac{d_p(\beta_p + 2\gamma_p) + d_{p-1}\delta_{p-1}}{2d_p\alpha_p}. \end{aligned}$$

Similarly, the remaining rows of $\mathcal{K}(d,\varepsilon)$ provide bounds on ζ_i for $i = 1,\ldots,p-1$ as

$$\begin{aligned} \frac{1}{\varepsilon_{i-1}}d_i\left(\alpha_i\zeta_i - \gamma_i\right) &> \frac{1}{2\varepsilon_{i-1}}\left(d_i\beta_i + d_{i-1}\delta_{i-1}\right) \\ &\quad + \frac{1}{2\varepsilon_i}\left(d_{i+1}\beta_{i+1} + d_i\delta_i\right) \\ \frac{1}{\varepsilon_{i-1}}d_i\left(\alpha_i\zeta_i - \gamma_i\right) &> \frac{1}{2\varepsilon_{i-1}}\left(d_i\beta_i + d_{i-1}\delta_{i-1}\right) \\ &\quad + \frac{\zeta_i}{2\varepsilon_{i-1}}\left(d_{i+1}\beta_{i+1} + d_i\delta_i\right) \\ 2d_i\left(\alpha_i\zeta_i - \gamma_i\right) &> \left(d_i\beta_i + d_{i-1}\delta_{i-1}\right) \\ &\quad + \zeta_i\left(d_{i+1}\beta_{i+1} + d_i\delta_i\right) \\ \zeta_i &> \frac{d_i(\beta_i + 2\gamma_i) + d_{i-1}\delta_{i-1}}{d_i(2\alpha_i - \delta_i) - d_{i+1}\beta_{i+1}}. \end{aligned}$$

■

The bounds on $\zeta_i = \varepsilon_i/\varepsilon_{i+1}$ in Theorem 3.6 describes a multiplicative factor between time-scales, i.e, the dynamics of layer $i+1$ is faster than the dynamics of layer i by a factor of ζ_i. For the ordering of the time-scales to be preserved as $\varepsilon_0 > \varepsilon_1 > \cdots > \varepsilon_p$, ζ_i must be at least 1.

The computational benefit of Theorem 3.6 over the aforementioned optimization problem (7) comes at the price of conservativeness. The minimum ζ_i from the bounds in Theorem 3.6 will be at least as large as the solution for optimization (7) under a convex cost function such as $q(\varepsilon_0^{-1},\varepsilon_1^{-1},\ldots,\varepsilon_p^{-1}) = \sum_{i=0}^{p}\frac{1}{\varepsilon_i}$.

IV. Network-of-Networks over State Dependent Graphs

The dynamics in Section III are general time-scale layered dynamics with no details of the underlying network structure. This section examines a special class of the dynamics (3) where the network explicitly manifests itself. The considered dynamics are a modification of the popular consensus dynamics running within each layer i over a graph $\mathcal{G}_i$. The graph $\mathcal{G}_i$ defines the direct coupling between node states x_i within layer i.

In many practical systems the topology of the layer $\mathcal{G}_i$ is dependent on the inter-network state and informed by outer-network states. This graph is assumed to be state dependent with $\mathcal{G}_i = \left(V_i, E_i, W_i(x_{i-1}, D(\mathcal{G}_i)^T x_i)\right)$, where the weights of the graph are dependent on the relative node states $D(\mathcal{G}_i)^T x_i$ within layer i and the states within the closest slower layer $i-1$. Consequently, the graph $\mathcal{G}_i$ evolves with the state x_i of its network as is influenced by the slowly changing states of its neighboring network.

As well as performing consensus over the graph $\mathcal{G}_i$, the dynamics of layer i are also influenced by the fastest adjacent layer states x_{i+1} through a function $f_i(\cdot)$. The function $f_i(\cdot)$ is also dependent on the edge states of the layer i, namely $D(\mathcal{G}_i)^T x_i$. The augmented modified consensus dynamics of layer i is[1]

$$\varepsilon_i\dot{x}_i = -L(\mathcal{G}_i)x_i + D(\mathcal{G}_i)f_i(D(\mathcal{G}_i)^T x_i, x_{i+1}), \qquad (10)$$

where $f_i(\cdot) = \mathbb{R}^{|E_i|\times 1}$. Similar to the traditional consensus, dynamics (10) is an edge-based dynamics, requiring only information about relative measurements of node states in the graph $\mathcal{G}_i$. A consequence of this formulation is that the agreement subspace is invariant as

$$\mathbf{1}^T\dot{z}_i = -\mathbf{1}^T L(\mathcal{G}_i)z_i + \mathbf{1}^T D(\mathcal{G}_i)f_i(D(\mathcal{G}_i)^T x_i, x_{i+1}) = 0.$$

The dynamics can thus be reparametrized with respect to the edge states $z_i = Cx_i \in \mathbb{R}^{n_i}$ and the static agreement state $\bar{z}_i = \frac{1}{n_i}\mathbf{1}^T x_i \in \mathbb{R}$ of the node dynamics x_i, where C forms an orthonormal basis of $\mathbb{R}^n\backslash\mathbf{1}$. The dynamics are equivalent to[2]

$$\varepsilon_i\dot{z}_i = -L_e(\mathcal{G}_i)z_i + f_{ei}(z_i, z_{i+1}; \bar{z}_{i+1}) := g_i(z_{i-1}, z_i, z_{i+1}) \qquad (11)$$

where $\dot{\bar{z}}_{i+1} = 0$, $\mathcal{G}_i = (V_i, E_i, W_i(z_{i-1}, z_i; \bar{z}_{i-1}))$, $L_e(\mathcal{G}_i) = CL(\mathcal{G}_i)C^T$ and $f_{ei}(z_i, z_{i+1}; \bar{z}_{i+1}) = CD(\mathcal{G}_i)f_i(D(\mathcal{G}_i)^T C^T z_i, C^T z_{i+1}; \bar{z}_{i+1})$. The origin for dynamics (11) is associated with each layer reaching agreement, or consensus, on its state. The following are two assumptions on the functions in dynamics (11) that mirror the assumptions in Section III.

Assumption 4.1. *Each function $L_e(\mathcal{G}_i)$ is continuously differentiable with respect to z_{i-1}, z_i, and z_{i+1} over its respective domains $D_{i-1} \subset \mathbb{R}^{n_{i-1}}$, $D_i \subset \mathbb{R}^{n_i}$ and $D_{i+1} \subset \mathbb{R}^{n_{i+1}}$ where $\bar{z}_{i-1} \in \bar{D}_{i-1} \subset \mathbb{R}$ $\bar{z}_{i+1} \in \bar{D}_{i+1} \subset \mathbb{R}$. Further, $f_{ei}(z_i, z_{i+1}; \bar{z}_{i+1})$ is Lipschitz continuous with $\|f_{ei}(x_1, y_1; \bar{z}_{i+1}) - f_{ei}(x_2, y_2; \bar{z}_{i+1})\| \leq \kappa_i(\bar{z}_{i+1})\left\|\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} - \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right\|$ over its respective domain, and $f_{ei}(z_i, z_{i+1}; \bar{z}_{i+1})$ has a unique, isolated zero $(z_i, z_{i+1}) = (0, 0)$.*

[1] As in Section III, $x_{-1}(t) = 0$ and $x_{p+1}(t) = 0$ are dummy states introduced to aid in clarity.

[2] Here, the notation $f(a; b)$ denotes that the parameter a is variable and the parameter b is constant.

Assumption 4.2. *The graph $\mathcal{G}_i$ is connected for all $z_{i-1} \in D_{i-1}$, $z_i \in D_i$ and $\bar{z}_{i-1} \in \bar{D}_{i-1}$.*

Assumption 4.1 is similar to Assumption 3.1 with the Lipschitz continuity of $f_{ei}(\cdot)$ replacing differentiability and the Lipschitz constant explicitly stated as $\kappa_i(\bar{z}_{i+1})$. Assumption 4.2 ensures that the dynamics $\dot{z}_i = -L_e(\mathcal{G}_i)z_i$ is asymptotically stable uniformly for all values of z_{i-1} in its domain, similar to Assumption 3.2.

A. Separation Principle

The dynamics (11) satisfies Assumption 4.1 of the previous section and so the separation principle is potentially applicable. The following proposition describes conditions on $f_{ei}(z_i, z_{i+1}; \bar{z}_{i+1})$ so that Assumption 3.2 holds.

Proposition 4.3. *Let $\underline{\lambda}_2(\mathcal{G}_i) = \min_{D_{i-1}\times D_i} \lambda_2(\mathcal{G}_i(z_{i-1}, z_i))$ with Assumptions 4.1 and 4.2 in place. If $\underline{\lambda}_2(\mathcal{G}_i) > \kappa_i(\bar{z}_{i+1})$ for all $z_{i-1} \in D_{i-1}$, $z_i \in D_i$ and $\bar{z}_{i+1} \in \overline{D}_{i+1}$, then $V_i(z_{i-1}, z_i) = \frac{1}{2}\lambda_2(\mathcal{G}_i(z_{i-1}, 0))z_i^T z_i$ is a positive-definite Lyapunov function such that*

$$\frac{V_i(z_{i-1}, z_i)}{\partial z_i} g_i(z_{i-1}, z_i, 0) \leq -\alpha_i \psi_i^2(z_i),$$

with $\alpha_i = \underline{\lambda}_2(\mathcal{G}_i)\left(\underline{\lambda}_2(\mathcal{G}_i) - \kappa_i(\bar{z}_{i+1})\right) > 0$ and $\psi_i(z_i) = \|z_i\|_2$.

Proof: Examining the derivative of $V_i(z_{i-1}, z_i)$ with respect to z_i then

$$\frac{\partial V_i(z_{i-1}, z_i)}{\partial z_i} = \lambda_2(\mathcal{G}_i(z_{i-1}, 0))z_i^T.$$

Using the fact that $\lambda_j(L_e(\mathcal{G}_i)) = \lambda_{j+1}(L(\mathcal{G}_i))$ for $j = 1, \ldots, n_i$ and the property that $f_{ei}(z_i, 0; \bar{z}_{i+1})$ is Lipschitz continuous, then

$$\begin{aligned}
&\frac{\partial V_i}{\partial z_i} g_i(z_{i-1}, z_i, 0) \\
&= \lambda_2(\mathcal{G}_i(z_{i-1}, 0))z_i^T \left(-L_e(\mathcal{G}_i)z_i + f_{ei}(z_i, 0; \bar{z}_{i+1})\right) \\
&\leq -\lambda_2(\mathcal{G}_i(z_{i-1}, 0)) \\
&\quad \times (\lambda_2(\mathcal{G}_i(z_{i-1}, z_i)) \|z_i\|_2^2 - \kappa_i(\bar{z}_{i+1}) \|z_i\|_2^2) \\
&\leq -\lambda_2(\mathcal{G}_i(z_{i-1}, 0)) \left(\lambda_2(\mathcal{G}_i(z_{i-1}, z_i)) - \kappa_i(\bar{z}_{i+1})\right) \|z_i\|_2^2 \\
&\leq -\underline{\lambda}_2(\mathcal{G}_i) \left(\underline{\lambda}_2(\mathcal{G}_i) - \kappa_i(\bar{z}_{i+1})\right) \|z_i\|_2^2.
\end{aligned}$$

As $\mathcal{G}_i$ is connected over the domain by Assumption 3.1 then $\underline{\lambda}_2(\mathcal{G}_i) > 0$ and the result follows. ∎

Proposition 4.3 provides conditions for the origin of the ith layer of dynamics to be asymptotically stable uniformly for all values of z_{i-1} for z_{i+1} fixed at the origin. The condition $\underline{\lambda}_2(\mathcal{G}_i) > \kappa_i(\bar{z}_{i+1})$ indicates that the worst case convergence rate of the consensus dynamics running on $\mathcal{G}_i$ must be greater than the rate of change of $f_{ei}(\cdot)$. Theorem 3.4 can now be directly applied to dynamics (11) describing its relationship to the ith reduced dynamics (4).

Proposition 4.4. *Under the dynamics (11) and with Assumptions 4.1, 4.2 and the condition in Proposition 4.3 satisfied, there exists an $\varepsilon_i^* > 0$, such that for $0 \leq \varepsilon_i \leq \varepsilon_i^*$ then for the ith reduced dynamics (4) then $z_i^{(i)} - z_i = O(\varepsilon_i)$ holds uniformly for some interval $t \in [t_1, t_2]$ where $0 < t_1 < t_2$.*

B. Bounds on ε_i

Similar to Section III-B, the Lyapunov functions associated with each layer of the network can be composed to analyze the full-order dynamics (11). This approach employs Lyapunov's direct method to obtain sufficient conditions on the time-scale parameters $\varepsilon_0, \ldots, \varepsilon_p$ so the full-order dynamics is asymptotically stable. Theorem 3.5 is applied to dynamics (11) to obtain the following result, with the associated proof inspired by a similar Lyapunov construction as in [11].

Proposition 4.5. *Under the dynamics (11) and with Assumptions 4.1, 4.2 and the condition in Proposition 4.3 satisfied, if there exists constants $\gamma_i \geq 0$ for all $i = 1, \ldots, p$ such that*

$$\mathbf{1}^T \frac{\partial w_i}{\partial z_{i-1}} g_{i-1}(z_{i-2}, z_{i-1}, z_i) \leq \gamma_i$$

for all $(z_0, z_1, \ldots, z_p) \in D_0 \times D_1 \times \cdots \times D_p$, then the origin, representing the consensus subspace in each network layer, is asymptotically stable for all $\varepsilon = [\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_p]^T > 0$, such that $\mathcal{K}(d, \varepsilon)$ in (5), is positive definite for some $d = [d_0, d_1, \ldots, d_p]^T$, where $\sum_{i=0}^p d_i = 1$ and $d_i \in (0, 1)$ for all i, α_i is defined in Proposition 4.3, $\beta_i = 0$ for all $i = 1, \ldots, p$, and $\delta_i = \underline{\lambda}_2(\mathcal{G}_i)\kappa_i(\bar{z}_{i+1})$ for all $i = 0, \ldots, p-1$.

Proof: Applying Theorem 3.5 to dynamics (11) then from Proposition 4.3 $\psi_i(z_i) = \|z_i\|_2$. Condition (2) in Theorem 3.5 for $V_i(z_{i-1}, z_i)$ defined in Proposition 4.3 simplifies to

$$\begin{aligned}
&\frac{\partial V_i}{\partial z_i} \left(g_i(z_{i-1}, z_i, z_{i+1}) - g_i(z_{i-1}, z_i, 0)\right) \\
&= \lambda_2(\mathcal{G}_i(z_{i-1}, 0))z_i^T \\
&\quad \times \left(f_{ei}(z_i, z_{i+1}; \bar{z}_{i+1}) - f_{ei}(z_i, 0; \bar{z}_{i+1})\right) \\
&\leq \lambda_2(\mathcal{G}_i(z_{i-1}, 0))\kappa_i(\bar{z}_{i+1}) \|z_i\|_2 \|z_{i+1}\|_2 \\
&\leq \underline{\lambda}_2(\mathcal{G}_i)\kappa_i(\bar{z}_{i+1}) \|z_i\|_2 \|z_{i+1}\|_2 \\
&= \delta_i \psi_i(z_i)\psi_i(z_{i+1}).
\end{aligned}$$

With regard to condition (1) in Theorem 3.5

$$\begin{aligned}
\frac{\partial V_i}{\partial z_{i-1}} &= \frac{1}{2}\frac{\partial \lambda_2(\mathcal{G}_i(z_{i-1}, 0)}{\partial z_{i-1}} \|z_i\|_2^2 \\
&= \frac{1}{2}\frac{\partial \lambda_2(\mathcal{G}_i(z_{i-1}, 0)}{\partial w_i}\frac{\partial w_i}{\partial z_{i-1}} \|z_i\|_2^2,
\end{aligned}$$

and using the property that $\left\|\frac{1}{2}\frac{\partial \lambda_2(\mathcal{G}_i(z_{i-1}, 0)}{\partial w_i}\right\|_\infty \leq 1$ [21], then

$$\begin{aligned}
&\frac{\partial V_i}{\partial z_{i-1}} g_{i-1}(z_{i-2}, z_{i-1}, z_i) \\
&= \frac{1}{2}\frac{\partial \lambda_2(\mathcal{G}_i(z_{i-1}, 0)}{\partial w_i}\frac{\partial w_i}{\partial z_{i-1}} \|z_i\|_2^2 g_{i-1}(z_{i-2}, z_{i-1}, z_i) \\
&\leq \mathbf{1}^T \frac{\partial w_i}{\partial z_{i-1}} g_{i-1}(z_{i-2}, z_{i-1}, z_i) \|z_i\|_2^2 \\
&\leq \gamma_i \psi_i^2(z_i).
\end{aligned}$$

∎

Proposition 4.5 provides a sufficient condition via a positive definite matrix condition for each network layer to converge to agreement. This condition is dependent on α_i, δ_i and γ_i, for each layer i. Smaller time-scale separation is required with larger α_i, and smaller δ_i and γ_i. This corresponds to larger $\underline{\lambda}_2(\mathcal{G}_i)$ (defined in Proposition 4.3) which is the slowest rate of convergence of the consensus dynamics over the domain. Further, smaller $\kappa_i(\bar{z}_{i+1})$ allows more flexibility in the time-scale parameter selection. This Lipschitz parameter encodes the deviation of the dynamics of layer i from traditional consensus due to the intra-layer coupling term $f_{ie}(\cdot)$. Finally, γ_i bounds the impact of the intra-layer dynamics on the state-dependent graph $\mathcal{G}_i$.

Proposition 4.5 also describes matrix $\mathcal{K}(d,\varepsilon)$ which can be optimized to select favorable ε and provide approximations of the domain of attraction. The results in Section III-C regarding these optimizations can be directly applied to the dynamics (11) via $\mathcal{K}(d,\varepsilon)$.

V. Examples

Consider a human-UAV swarm network-of-networks problem composed of three time-scale separated layers. The fastest layer's dynamics, with time-scale ε_2, is performing altitude rendezvous over a swarm of n UAVs using relative sensing over their altitude state $x_2 \in \mathbb{R}^{n_v}$. The sensing is performed over a state dependent graph $\mathcal{G}_2(x_1, x_2)$, dependent on the relative altitude of the vehicles and the power expense x_1 of sensing. For the moderate speed dynamics, with time-scale ε_1, its purpose is to reach agreement on agent power usage $x_1 \in \mathbb{R}^{n_v}$ negotiating over network communication and with the aid of limited GPS. The communication is performed over a state-dependent graph $\mathcal{G}_1(x_0)$ with communication rate dictated by a human opinion state x_0. The slowest network layer, with time-scale ε_0, involves n_h humans monitoring disjoint subsets $S = (S_1, \ldots, S_h)$ of UAVs power requirements using a human opinion state $x_0 \in \mathbb{R}^{n_h}$. The human-interaction network is a static graph $\mathcal{G}_0$. The generalized dynamics is

$$\begin{aligned} \varepsilon_0 \dot{x}_0 &= -L(\mathcal{G}_0)x_0 + f_0(x_1) \\ \varepsilon_1 \dot{x}_1 &= -L(\mathcal{G}_1(x_0))x_1 + f_1(x_2) \\ \varepsilon_2 \dot{x}_2 &= -L(\mathcal{G}_2(x_1, x_2))x_2. \end{aligned} \tag{12}$$

The function $[f_0(x_1)]_i$ encodes the ith human's monitoring of its S_i UAVs by comparing the power x_1 of its agents against the average $\bar{x}_1$ of the swarm's power formulated as

$$[f_0(x_1)]_i = c_h \sum_{j \in S_i} \left(\bar{x}_1 - [x_1]_j \right),$$

or alternatively as

$$f_0(x_1) = c_h P(S)(\frac{1}{n_v}\mathbf{1}\mathbf{1}^T - I)x_1,$$

where the ith row of the matrix $P(S) \in \mathbb{R}^{n_v \times n_h}$ is an indicator vector of S_i which is the UAVs that the ith human is monitoring and c_h is a positive scale factor.

The function $f_1(x_2)$ corresponds to the limited GPS information which informs the power consumption dynamics of the network. GPS is used to calculate the altitude distance between the first and second UAV $|[x_2]_2 - [x_2]_1|$ which updates the power usage $[x_1]_1$ and $[x_1]_2$. The GPS function is

$$f_1(x_2) = c_s(e_2 - e_1) \left| (e_2 - e_1)^T x_2 \right|,$$

where c_s is a positive scale factor.

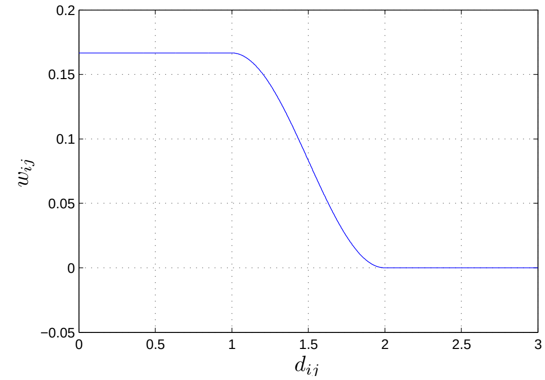


Figure 2. Plot of w_{ij} where $n_v = 6$ and $t_{ij} = 1$.

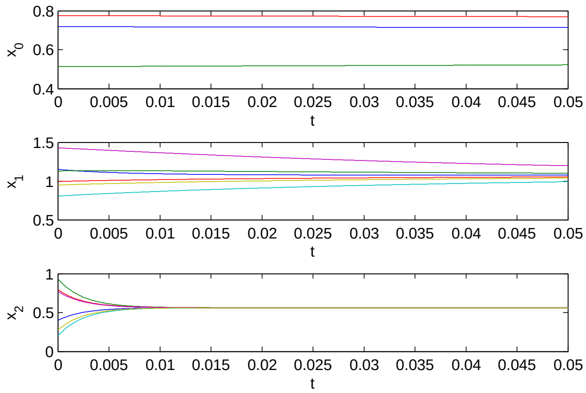
The communication graph $\mathcal{G}_1(x_0)$ is throttled based on the first human's opinion state, specifically every edge in $\mathcal{G}_1$ has weight $[x_0]_1$. The sensing layer has a state-dependent homophily graph $\mathcal{G}_2(x_1, x_2)$ with weights $w_{ij}(d_{ij}, t_{ij})$ where the weight on the edge is proportional to the altitude variation between the UAVs at either end of the edge $d_{ij} = \left| [x_2]_i - [x_2]_j \right| = \left| (e_i - e_j)^T x_2 \right|$ and thresholded using the power consumption restriction dictated using the edge ends power states as $t_{ij} = [x_1]_i + [x_1]_j = (e_i + e_j)^T x_1$. The specific homophily weight function is

$$w_{ij} = \begin{cases} \frac{1}{n_v} & d_{ij} \leq t_{ij} \\ \frac{1}{2n_v}(1 + \cos(\pi(d_{ij}/t_{ij} - 1))) & t_{ij} \leq d_{ij} \leq 2t_{ij} \\ 0 & \text{otherwise} \end{cases}$$

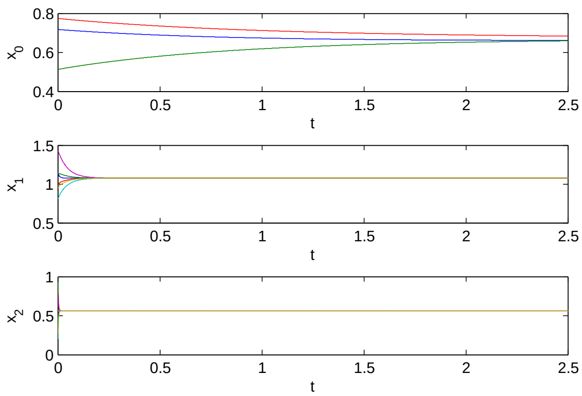
and displayed in Figure 2.

Dynamics (12) are in the form of (10) and so the results of Section IV are applicable. The relevant parameters in Theorem 4.5 are $\kappa_0(\bar{z}_0) = c_h \left\| P(S)(\frac{1}{n_v}\mathbf{1}\mathbf{1}^T - I) \right\|_2 \leq 2c_h\sqrt{\max_i |S_i|}$, $\kappa_1(\bar{z}_1) = c_s \left\| (e_2 - e_1)(e_2 - e_1)^T \right\|_2 = 2c_s$, $\kappa_3(\bar{z}_3) = \gamma_0 = \gamma_2 = 0$ and $\gamma_1 = \frac{1}{2}\left(\operatorname{diam}(D_0) + |S_1| \operatorname{diam}(D_1)\right)$.

Let $n_v = 6$, $n_h = 3$, $c_h = c_s = 1/6$, with $S = (\{1,2\}, \{3,4\}, \{5,6\})$. Let the underlying graphs $\mathcal{G}_0$, $\mathcal{G}_1$ and $\mathcal{G}_2$ in the problem be a path graph, a star graph and a complete graph, respectively. The domains in the problem are $D_0 = [0,1]^{n_h}$, $D_1 = [1,2]^{n_v}$ and $D_2 \in [0,1]^{n_v}$. Hence, $\underline{\lambda}_2(\mathcal{G}_0) = 1$, $\underline{\lambda}_2(\mathcal{G}_1) = 0.5$, $\min t_{ij} = 1$, $\max d_{ij} = 1.5$, $\min w_{ij}(d_{ij}, t_{ij}) = \frac{1}{2} + \frac{1}{2}\cos(\pi(1.5 - 1) = 1/12$, $\underline{\lambda}_2(\mathcal{G}_1) = 0.5$, $\kappa_0(\bar{z}_0) \approx 0.47$, $\kappa_0(\bar{z}_0) = 1/3$, $\gamma_1 = 1.75$, $\delta_0 = 2.8$,



(a)



(b)

Figure 3. Trajectory plot of the human-UAV swarm dynamics (12) with snapshots over (a) $t = [0, 0.05]$ and (b) $t = [0, 2.5]$.

$\delta_1 = 1$, $\delta_2 = 0$, $\alpha_0 \approx 0.53$, $\alpha_1 \approx 0.08$, $\alpha_2 = 0.25$. Therefore,

$$\kappa(d,\varepsilon) = \begin{bmatrix} 0.53d_0 & -1.4\frac{d_0}{\varepsilon_0} & 0 \\ -1.4\frac{d_0}{\varepsilon_0} & d_1(\frac{0.53}{\epsilon_1} - \frac{1.75}{\varepsilon_0}) & -\frac{1}{2}\frac{d_1}{\varepsilon_1} \\ 0 & -\frac{1}{2}\frac{d_1}{\varepsilon_1} & d_2\frac{0.25}{\varepsilon_p} \end{bmatrix}$$

and for $d = (0.2, 0.4, 0.4)$ a feasible set of time-scale parameters is $\varepsilon = (2.6, 0.032, 0.015)$, calculated using the bounds in Theorem 3.6. The resultant dynamics converge to agreement in each of the network states. The trajectories are displayed with the final time $t = 0.05$ and $t = 2.5$ in Figure 3, respectively.

VI. Conclusion

In this work, a network-of-networks dynamic system was explored by exploiting the inherent time-scale separation in the problem. Analysis of a general form of the problem provided a bilinear matrix inequality condition to guarantee sufficient time-scale separation between network layers for asymptotic stability of the composite system. Techniques from convex optimization provided a mechanism to find sufficient time-scale parameters to satisfy this inequality condition. This framework was applied to a multi-layered state-dependent network running a modified consensus dynamics. Links to the underlying network topology and the aforementioned inequality condition were established providing a graph-theoretic measure of time-scale separation stability. A human-UAV swarm example demonstrated the utility of the results. Future work will leverage the optimization approach presented here coupled with network synthesis techniques to design network-of-networks dynamics that stabilized through their network topology.

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