State Controllability, Output Controllability and Stabilizability of Networks: A Symmetry Perspective

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The Network in the Dynamics

General Dynamics

\[
\dot{x}(t) = f(G, x(t), u(t))
\]
\[
y(t) = g(G, x(t), u(t))
\]

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The Network in the Dynamics

\[ \dot{x}(t) = -A(G)x(t) + B(S)u(t) \]
\[ y(t) = C(R)x(t) \]

- First Order, Linear Time Invariant model
- e.g., Laplacian, Adjacency, Advection matrices
- Input node set \( S = \{v_i, v_j, \ldots\} \), \( B(S) = [e_i, e_j, \ldots] \)
- Output node set \( R = \{v_p, v_q, \ldots\} \), \( C(R) = [e_p, e_q, \ldots]^T \)
Network Controllability

- Given a controllable/observable linear system we can
  - guarantee existence of a stabilizing controller using feedback and observer techniques
  - with augmented dynamics we can achieve steady state tracking and disturbance/noise rejection
  - reason about modes that can be manipulated from the input or observed from the output

- General area: **Structural controllability** (Liu et al. ’11), **strong controllability** (Jarczyk et al. ’11), **nonlinear controllability** (Aguilar and Bahman ’14), **degree of controllability** (Chapman et al. ’12)

- Specific area: **Graph Families** (Parlengeli et al. ’11, Zhang et al. ’11), **Symmetry** (Rahmani and Mesbahi ’06), **Equitable Partitions** (Martini et al. ’10), **Graph distances** (Yazicioglu and Bahman ’12), **Zero forcing sets** (Burgarth et al. ’14)
Proposition (Rahmani and Mesbahi 2006)

\((A(\mathcal{G}), B(S))\) is uncontrollable if there exists an automorphism of \(\mathcal{G}\) which fixes all inputs in the set \(S\).

The determining number of a graph \(\mathcal{G}\), denoted \(\text{Det}(\mathcal{G})\), is the smallest integer \(r\) so that \(\mathcal{G}\) has a determining set \(S\) of size \(r\).

Cardinality requirement: If the pair\((A(\mathcal{G}), B(S))\) is controllable then \(|S| \geq \text{Det}(\mathcal{G})\).
Necessary and Sufficient Condition?

Example

For the asymmetric graph $G$, $(-L(G), B(S))$ uncontrollable, where $S = \{1\}$. (Rahmani et al. ’06)

- How to analyze this $G$?

... signed fractional graph automorphisms
Fractional graph automorphisms

- Algebraic condition on $P$ to represent an automorphism of the graph $G$ is
  \[ A(G)P = PA(G) \]
  \[ 1^T P = 1^T \]
  \[ P1 = 1 \]
  \[ P_{ij} \in \{0, 1\}. \]

- Similarly, $P$ represents an automorphism with respect to $L(G)$ exists if condition (1) is replaced with $L(G)P = PL(G)$

- **Fractional automorphism**: Scheinerman and Ullman proposed an integer relaxation of condition (2) to $P_{ij} \geq 0$
  - $P$ is doubly stochastic matrix instead of a permutation matrix
  - **Signed fractional automorphism**: Relaxed further to positive and nonpositive $P_{ij} \in \mathbb{R}$

**Other Features**
- The set of $P$’s forms a convex polytope
- $P$ has a relaxed perfect matching interpretation
- $G$ has no non-trivial fractional automorphisms $\iff G$ has no nontrivial equitable partitions
Fractional Input Symmetry

- A nontrivial signed fractional automorphism (SFA) with respect to $L(G)$ is input symmetric if it fixes the input node set $S$.
- $S$ is referred to as fractional input symmetric.
- Equivalently, SFA matrix representation $P \neq I$ with $PB(S) = B(S)$ is referred to as input symmetric.

**Theorem**

*(Chapman and Mesbahi ’14)* The pair $(-L(G), B(S))$ is uncontrollable $\iff$ there exists an input symmetric nontrivial SFA ($S$ is fractional leader symmetric).

**Example**

Input set $S = \{1\}$ is uncontrollable as $S$ is fractional leader symmetric with associated

$$P = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{5} \left( \mathbf{11}^T - I \right) \end{bmatrix}.$$
Diagonalizable Digraph Extension

Theorem

For diagonalizable $L(D)$, the pair $(-L(D), B(S))$ is uncontrollable $\iff$ there exists an input symmetric nontrivial SFA ($S$ is fractional input symmetric)

Examples of Diagonalizable Digraphs

- Acyclic digraphs: contain exactly one rooted spanning tree
- Simple Laplacian digraphs: Laplacian has no repeated eigenvalues, e.g., special families of directed digraph tournaments (Caen et al. ’92)
- Normal Laplacian digraphs: $L(G)L(G)^T = L(G)^T L(G)$, e.g., Cayley digraphs (Lyubshin and Savchenko ’09)
- Strongly Regular digraphs: the number of paths of length 2 starting at $i$ and finishing $j$ is $t$ if $i = j$, $\lambda$ if $(i,j) \in E$ and $\mu$ otherwise.

Example

A smallest controllable input set is $S = \{1,2\}$ with $S$ fractional leader asymmetric.
Output Controllability

Output controllability indicates if the output nodes $R$ are controllable from input nodes $S$.

**Theorem**

For diagonalizable $L(D)$, the triple $(-L(D), B(S), C(R))$ is output uncontrollable if there exists a nontrivial SFA which is simultaneously input symmetric, and non-output $V \setminus R$ symmetric.

Intuition: Nontrivial non-output symmetry forces output asymmetry. (NB: Theorem has been corrected from the paper to an sufficient condition only)

**Example**

The system is output uncontrollable with $R_2 = \{2, 3\}$ and non-output $V \setminus R_2 = \{1, 4\}$ with

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix}.$$
Stabilizability

- Recent interest in negatively weighted graphs which can induce unstable modes (Zelazo and Burger '14)

Stabilizability indicates if the unstable modes are controllable from input nodes $S$.
- A stable symmetric SFA satisfies $P \preceq I$ and $Pv_i = v_i$ where $\lambda_i(L(G)) > 0$.

**Theorem**

The pair $(-L(G), B(S))$ is output unstabilizable $\iff$ there exists a nontrivial SFA which is simultaneously input symmetric and **stable symmetric**.

**Intuition:** Nontrivial stable symmetry forces **unstable asymmetry**.

**Example**

The system is stabilizable from input $S_1 = \{1\}$ but not $S_2 = \{2\}$ with

$$P = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$
Optimization Formulation

- Equivalent optimization formulation for the (output) controllable/stabilizable conditions:

\[
\min_{P \in \mathbb{R}^{n \times n}, Z \in \mathbb{R}^{n \times p}} \text{tr} P
\]

\[
L(G)P = PL(G), \; PB = B
\] (2)

\[
P1 = 1, \; P = P^T
\] (3)

\[
P = I - ZC
\] (4)

\[
P \preceq I, (P - I)L(G) \succeq 0
\] (5)

- Objective (1) prevents, if possible, the trivial automorphism \( P = I \) solution
- Lines 1-3 check for controllability of \((-L(G), B)\)
- Lines 1-3 and 4 checks for output controllability of the triple \((-L(G), B, C)\)
- Lines 1-3 and 5 checks for stabilizability of the pair \((-L(G), B)\)
- Lines 1-3, 4 and 5 checks for output stabilizability of the triple \((-L(G), B, C)\).
Conclusion

- Explored the link between symmetry and (output) controllability and stabilizability of networked dynamics
- Related using the notion of (signed) fractional automorphisms (SFA) - a relaxation of traditional graph automorphisms
- Provided necessary and sufficient conditions using (signed) fractionally input symmetry and unstable asymmetry
- Provided necessary conditions using (signed) fractionally input symmetry and output asymmetry
- Posed network (output) controllability/stabilizability as a convex optimization problem
- Future work involves coupling the controllability convex optimization problem with other network design requirements

Edge weights optimized for distance to uncontrollability from $S = \{2\}$ and network connectivity

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If all eigenvalues of $\mathcal{A}(G)$ are simple, then every automorphism of $G$ has order 1 or 2, i.e., $\sigma(\sigma(i)) = i$ for all $i \in V$.

Let $G = G_1^{k_1} \square \cdots \square G_m^{k_m}$ be the prime factor decomposition for a connected graph $G$. Then $\text{Det}(G) = \max\{\text{Det}(G_i^{k_i})\}$. (Boutin ’09)

Strongly $k$-regular graphs are asymmetric with high eigenvalue multiplicity, i.e., exactly three eigenvalues.

Circulants graph on a prime number are uncontrollable from $q$ consecutive inputs $\iff$ eigenvalue geometric multiplicity is greater than $q \iff$ leader symmetric (Nabi-Abdolyousefi et al. ’12)
Popov-Belevitch-Hautus (PBH) test

$(A, B)$ is uncontrollable if and only if there exists a left eigenvalue-eigenvector pair $(\lambda, \nu)$ of $A$ such that $\nu^T B = 0$

- The eigenvectors of $L$ are $\nu_1 = 1, \nu_2, \ldots,$ with eigenvalues $\lambda_1 = 0, \lambda_2, \ldots$

$(\implies)$

- $(L, B)$ uncontrollable, $1^T B \neq 0 \implies \nu_i^T B = 0$ for some $i \neq 1$. Let $P_i = I - \nu_i \nu_i^T \implies P_i B = (I - \nu_i \nu_i^T)B = B$.
- Further $LP_i = L - L \nu_i \nu_i^T = L - \lambda_i \nu_i \nu_i^T = L - \nu_i \nu_i^T L = P_i L$ and $P_i 1 = P_i^T 1 = 1$.

$(\impliedby)$

- For $PL = LP$, $P \neq I$, then $(I - P)w \neq 0$ is a left-eigenvector for some left-eigenvector pair $(\mu, w)$ as

$$w^T(I - P)L = w^T L(I - P) = \mu w^T(I - P)$$

- $PB = B \implies w^T(I - P)B = w^T B - w^T B = 0 \implies (L, B)$ uncontrollable
Matching Perspective

- Examine perfect matchings $\mathcal{E}$ between $G = (V, E)$ and $G' = (V', E')$.
- Construct graph $H = (V \cup V', E \cup E' \cup \mathcal{E})$: Refer to $E$ and $E'$ as edges and $\mathcal{E}$ as links.
- A matching corresponds to an automorphism if an edge-link path $i \rightarrow a \rightarrow j'$ exists iff a link-edge path $i \rightarrow b' \rightarrow j'$ exists.
- If $\mathcal{E}$ is now a (signed) fractional matching composed of weighted links.
- A matching corresponds to an (signed) fractional automorphism if the sum of weighted links of all edge-link paths from $i$ to $j'$ equals the sum of weighted links of all link-edge paths from $i$ to $j'$.
Compact Graphs

- Let $S(A(G))$ be all the doubly stochastic matrices that commute with $A(G)$, then $S(A(G))$ represents the fractional automorphisms of $G$ with respect to $A$.
- $\text{Aut}(G) = \{P_1, \ldots, P_t\} \subseteq S(A(G))$ for all permutation matrices $P_i$ representing the automorphisms, hence $\text{conv}(\text{Aut}(G)) \subseteq S(A(G))$.

**Definition**

$G$ is a **compact** graph $\iff \text{conv}(\text{Aut}(G)) = S(A(G))$

For compact graphs:
- $\text{Aut}(G)$ are all the extreme points of $S(A(G))$.
- The fractional automorphisms of $A$ and $L$ are shared.
- Known families: trees, cycles, complete, complement of compacts (if both connected), can be determined in polynomial time for prime node regular graphs.

**Proposition (Chan and Godsil 1997)**

For compact graphs, the automorphism problem is solved in polynomial time.
Equitable Partitions

**Definition**

An **equitable partition** of $\mathcal{G}$ is a partition $V$ into $\pi = \{C_1, C_2, \ldots, C_s\}$ s.t.
(a) vertices of $C_i$ induce a regular graph
(b) edges of $C_i$ and $C_j$ induce a half-regular graph

- Characteristic matrix $D$ of $\pi$ is an $n \times s$ matrix with $D_{ij} = 1$ if $i \in C_j$ and 0 otherwise
- Let $P = D(D^T D)^{-1} D^T$ then $P$ is symmetric and $P^2 = P$.

**Proposition (Godsil 1997)**

The partition $\pi$ is equitable $\iff$ $P$ represents a fractional automorphism

- Consequence: $\mathcal{G}$ has no non-trivial fractional automorphisms $\iff$ $\mathcal{G}$ has no nontrivial equitable partitions
Additional Features

- Every $k$-regular graph $G$ on $n$ nodes has a fractional automorphism represented by $P = \frac{1}{n} \mathbf{1}\mathbf{1}^T$. Further, $PA(G) = A(G)P \iff PL(G) = L(G)P$
  - The pair $(-L(G), B)$ is controllable if and only if $(A(G), B)$
  - The pair $(-L(G), B)$ is uncontrollable when $B = \alpha \mathbf{1}$ for all $\alpha \in \mathbb{R}$
- Equitable partitions have been previously linked to controllability
  - Uncontrollability is induced through equitable partitions $\iff$ fractional leader symmetric
  - Fractionally leader asymmetric for all inputs $\iff$ no non-trivial equitable partitions
- Fractional leader asymmetry is not sufficient for controllability

Counter-Example
Frucht graph with no fractional automorphisms is uncontrollable from single inputs $\{3, 4, 5, 7, 8, 9\}$. 

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