Cartesian Products of Z-Matrix Networks: Factorization and Interval Analysis

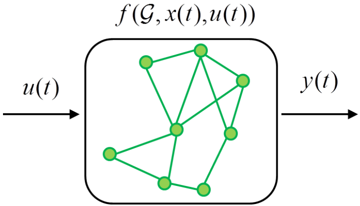
Airlie Chapman and Mehran Mesbahi
(work also in part with Marzieh Nabi-Abdolyousefi)

Distributed Space Systems Lab (DSSL)

University of Washington

General Dynamics

$$\dot{x}(t) = f(\mathcal{G}, x(t), u(t))$$
$$y(t) = g(\mathcal{G}, x(t), u(t))$$



Network	System Dynamics
Graph Spectrum	Rate of convergence
Random Graphs	Random Matrices
Automorphisms	Homogeneity
Graph Factorization	Decomposition

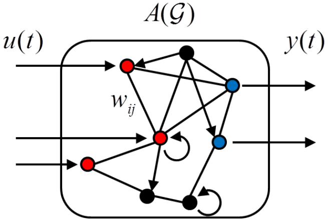
1. State Dynamics
2. Controllability

The Network in the Dynamics

- First Order, Linear Time Invariant model

$$\dot{x}_i(t) = -w_{ii}x_i(t) + \sum_{i\sim j} w_{ij}x_j(t) + u_i(t)$$

$$y_i(t) = x_i(t)$$



Dynamics

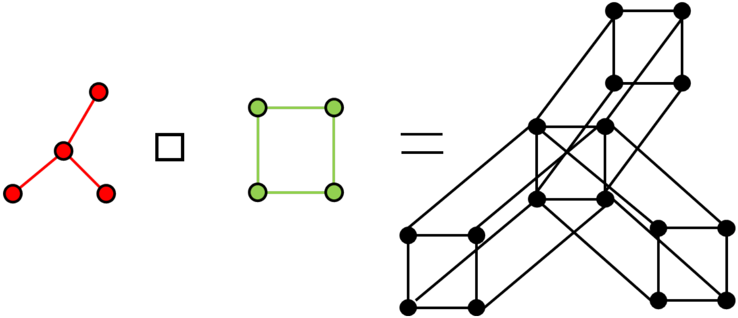
$$\dot{x}(t) = -A(\mathcal{G})x(t) + B(S)u(t)$$
$$y(t) = C(R)x(t)$$

$$A(\mathcal{G}) = \begin{bmatrix} w_{11} & -w_{12} & \cdots & -w_{1n} \\ -w_{21} & w_{22} & & \vdots \\ \vdots & & \ddots & -w_{n-1,n} \\ -w_{n1} & \cdots & -w_{n,n-1} & w_{nn} \end{bmatrix}$$

- $A(\mathcal{G})$: Z-matrix, e.g. Laplacian ($w_{ii} = \sum w_{ij}$) and Advection matrices ($w_{ii} = \sum w_{ji}$)
- Input node set $S = \{v_i, v_j, \dots\}$, $B(S) = [e_i, e_j, \dots]$
- Output node set $R = \{v_p, v_q, \dots\}$, $C(R) = [e_p, e_q, \dots]^T$

Graph Cartesian Product

- Cartesian product $\mathcal{G} \Box \mathcal{H}$
- Vertex set: $V(\mathcal{G} \Box \mathcal{H}) = V(\mathcal{G}) \times V(\mathcal{H})$
- Edge set: $(x_1, x_2) \sim (y_1, y_2)$ is in $\mathcal{G} \Box \mathcal{H}$
  - if $x_1 \sim y_1$ and $x_2 = y_2$ or $x_1 = y_1$ and $x_2 \sim y_2$



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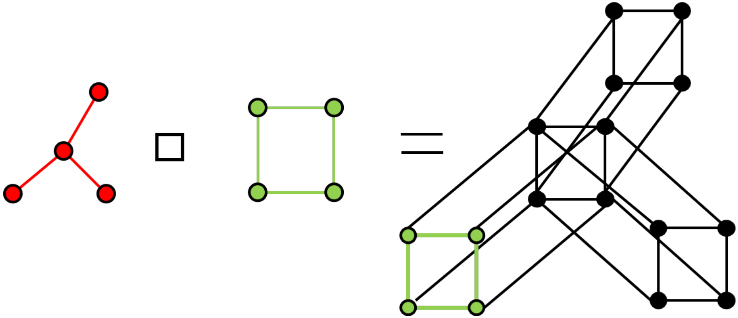

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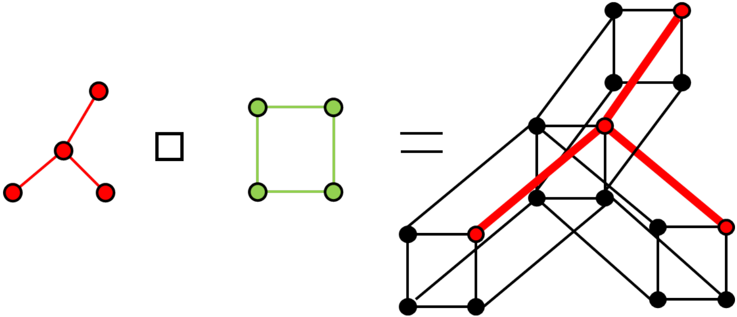


Part 1:
Factoring the State Dynamics

State Dynamics Factorization

Lemma 1: Uncontrolled Dynamics

Consider the dynamics

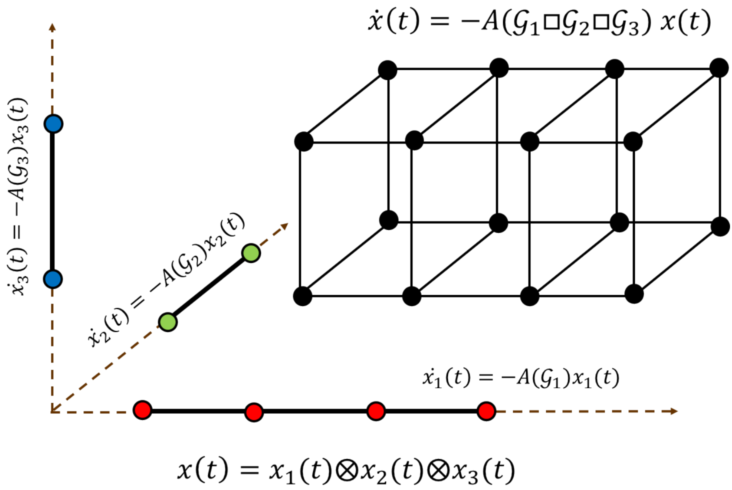
$$\dot{x}(t) = -A(\mathcal{G}_1 \square \mathcal{G}_2 \square \ldots \square \mathcal{G}_n)x(t).$$

Then, when properly initialized, one has

$$x(t) = x_1(t) \otimes x_2(t) \otimes \cdots \otimes x_n(t)$$

where $\dot{x}_i(t) = -A(\mathcal{G}_i)x_i(t)$.

State Dynamics Factorization



Graph Factorization

- A graph can be factored as well as composed...

Theorem (Sabidussi 1960)

Every connected graph can be factored as a Cartesian product of prime graphs. Moreover, such a factorization is unique up to reordering of the factors.

- Primes: $\mathcal{G} = \mathcal{G}_1 \Box \mathcal{G}_2$ implies that either $\mathcal{G}_1$ or $\mathcal{G}_2$ is K_1
  - Number of prime factors is at most $\log |\mathcal{G}|$
- Algorithms
  - Feigenbaum (1985) - $\mathcal{O}\left(|V|^{4.5}\right)$
  - Winkler (1987) - $\mathcal{O}\left(|V|^{4}\right)$ from isometrically embedding graphs by Graham and Winkler (1985)
  - Feder (1992) - $\mathcal{O}(|V||E|)$
  - Imrich and Peterin (2007) - $\mathcal{O}(|E|)$
  - C++ implementation by Hellmuth and Staude

Controlled Factorization Lemma

Lemma 2: Controlled Dynamics

Consider the dynamics

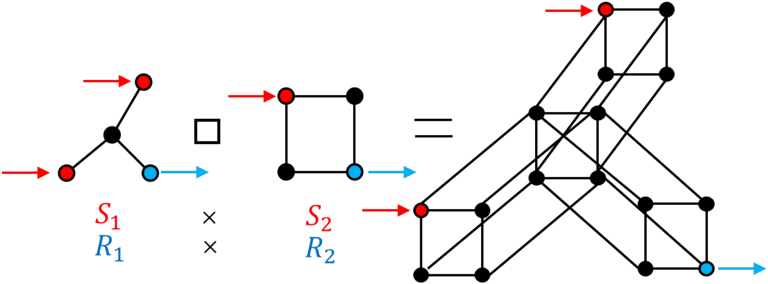
$$\dot{x}(t) = -A(\mathcal{G}_1 \square \mathcal{G}_2 \square \ldots \square \mathcal{G}_n)x(t) + B(S_1 \times S_2 \times \cdots \times S_n)u(t).$$
$$y(t) = C(R_1 \times R_2 \times \cdots \times R_n)x(t)$$

Then, when properly initialized and $u(t) = u_1(t) \otimes u_2(t) \otimes \cdots \otimes u_n(t)$, one has $y(t) = y_u(t) + y_f(t)$,

$$y_u(t) = y_{1u}(t) \otimes y_{2u}(t) \otimes \cdots \otimes y_{nu}(t)$$
$$y_f(t) = \int_0^t \dot{y}_{1f}(\tau) \otimes \dot{y}_{2f}(\tau) \otimes \cdots \otimes \dot{y}_{nf}(\tau) d\tau$$

where $\dot{x}_i(t) = -A(\mathcal{G}_i)x_i(t) + B(S_i)u_i(t)$ and $y_i(t) = C(R_i)x_i(t)$.

Controlled Factorization Lemma



Controlled Factorization Lemma

Lemma 2: Controlled Dynamics

Consider the dynamics

$$\dot{x}(t) = -A(\prod_{\square} \mathcal{G}_i)x(t) + B(\prod_{\times} S_i)u(t)$$
$$y(t) = C(\prod_{\times} R_i)x(t)$$

Then, when properly initialized and $u(t) = \prod_{\otimes} u_i(t)$, one has

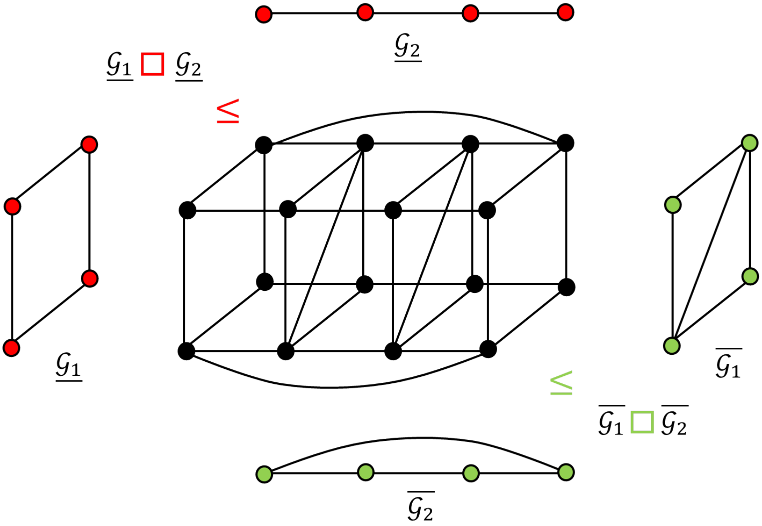
$$y(t) = \prod_{\otimes} y_{iu}(t) + \int_0^t \prod_{\otimes} \dot{y}_{if}(\tau) d\tau$$

where $\dot{x}_i(t) = -A(\mathcal{G}_i)x_i(t) + B(S_i)u_i(t)$ and $y_i(t) = C(R_i)x_i(t)$.

Controlled Factorization - Idea of the Proof

- Firstly
$$A(\mathcal{G}_1 \Box \mathcal{G}_2) = A(\mathcal{G}_1) \otimes I + I \otimes A(\mathcal{G}_2) = A(\mathcal{G}_1) \oplus A(\mathcal{G}_2)$$
- Also
$$e^{A(\mathcal{G}_1) \oplus A(\mathcal{G}_2)} = e^{A(\mathcal{G}_1)} \otimes e^{A(\mathcal{G}_2)}$$
- Using
$$B(S_1 \times S_2) = B(S_1) \otimes B(S_2)$$
- As well as
$$\begin{aligned} e^{-A(\mathcal{G}_1 \Box \mathcal{G}_2)t} Bu(t) &= (e^{-A(\mathcal{G}_1)t} \otimes e^{-A(\mathcal{G}_2)t})(B(S_1) \otimes B(S_2))(u_1(t) \otimes u_2(t)) \\ &= e^{-A(\mathcal{G}_1)t} B(S_1) u_1(t) \otimes e^{-A(\mathcal{G}_2)t} B(S_2) u_2(t) \end{aligned}$$
- The proof follows from these observations.

Approximate Graph Products



$$A(\mathcal{G}) \in \left[A(\underline{\mathcal{G}}_1 \square \underline{\mathcal{G}}_2), A(\overline{\mathcal{G}}_1 \square \overline{\mathcal{G}}_2)\right]$$

Approximate Factorization Lemma

Lemma 3: Approximation

Consider the dynamics

$$\dot{x}(t) = -A(\mathcal{G})x(t) + B(S)u(t),$$
$$y(t) = C(R)x(t)$$

$A(\mathcal{G}) \in \left[A(\prod_{\square} \underline{\mathcal{G}}_i), A(\prod_{\square} \overline{\mathcal{G}}_i)\right]$, $S \in \left[\prod_{\times} \underline{S}_i, \prod_{\times} \overline{S}_i\right]$, $R = \left[\prod_{\times} \underline{R}_i, \prod_{\times} \overline{R}_i\right]$ and positive $u(t) \in [\prod_{\otimes} \underline{u}_i(t), \prod_{\otimes} \overline{u}_i(t)]$. Then, when properly initialized, one has

$$\prod_{\otimes} \overline{y}_{iu}(t) + \int_0^t \prod_{\otimes} \dot{\overline{y}}_{if}(\tau)d\tau \leq y(t) \leq \prod_{\otimes} \underline{y}_{iu}(t) + \int_0^t \prod_{\otimes} \underline{\dot{y}}_{if}(\tau)d\tau$$

where $\dot{\underline{x}}_i(t) = -A(\underline{\mathcal{G}}_i)\underline{x}_i(t) + B(\underline{S}_i)\underline{u}_i(t)$, $\underline{y}_i(t) = C(\underline{R}_i)\underline{x}_i(t)$ and similarily for $\overline{y}_i(t)$.

Approximate Graph Products

- Can a graph be approximately factored as well as approximately composed?
- Distance between graphs $d(\mathcal{G},\mathcal{H})$ is the smallest integer k such that

$$\left|V(\mathcal{G}')\triangle V(\mathcal{H}')\right| + \left|E(\mathcal{G}')\triangle E(\mathcal{H}')\right| \le k.$$

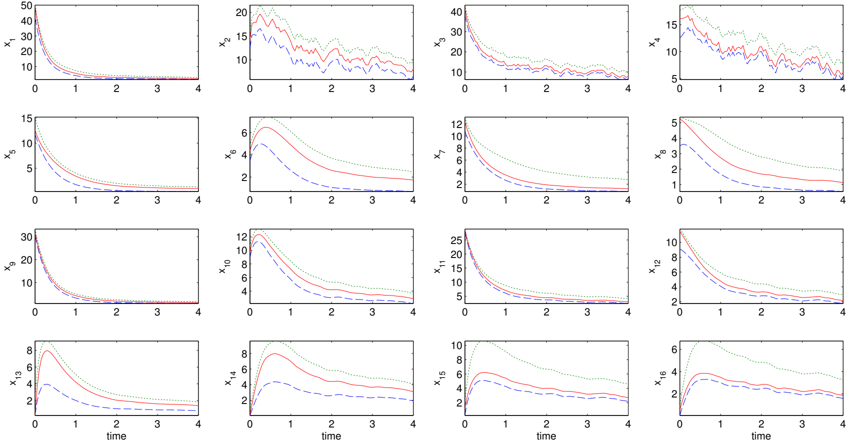
A graph $\mathcal{G}$ is a k-approximate graph product if there is a product $\mathcal{H}$ such that

$$d(\mathcal{G},\mathcal{H}) \le k.$$

Lemma (Hellmuth 2009)

For fixed k all Cartesian k-approximate graph products can be recognized in polynomial time.

Approximate Factorization Lemma: An Example



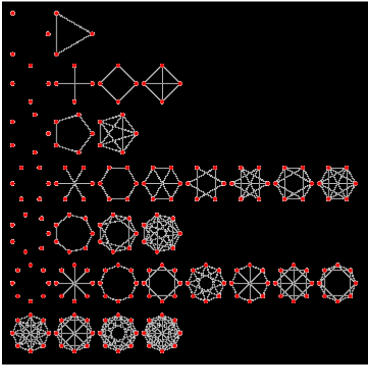
Part 2:

Factoring Controllability

(work with Marzieh Nabi-Abdolyousefi)

Controllability

- Dynamics are controllable if for any $x(0)$, x_f and t_f there exists an input $u(t)$ such that $x(t_f) = x_f$.
- Significant in networked robotic systems, human-swarm interaction, network security, quantum networks.
- Challenging to establish for large networks
- Known families of controllable graphs for selected inputs
    - Paths
    - Circulants
    - Grids
    - Distance regular graphs



Controllability Factorization - Product Control

Theorem 1: Product Controllability

The dynamics

$$\dot{x}(t) = -A(\prod_{\square} \mathcal{G}_i)x(t) + B(\prod_{\times} S_i)u(t)$$
$$y(t) = C(\prod_{\times} R_i)x(t)$$

where $A(\prod_{\square} \mathcal{G}_i)$ has simple eigenvalues is controllable/observable if and only if

$$\dot{x}_i(t) = -A(\mathcal{G}_i)x_i(t) + B(S_i)u_i(t)$$
$$y_i(t) = C(R_i)x_i(t)$$

is controllable/observable for all i.

Controllability Factorization - Idea of the Proof

Popov-Belevitch-Hautus (PBH) test

(A, B) is uncontrollable if and only if there exists a left eigenvalue-eigenvector pair (λ, v) of A such that $v^T B = 0$.

- Eigenvalue and eigenvector relationship:

	$A(\mathcal{G}_1)$	$A(\mathcal{G}_2)$	$A(\mathcal{G}_1 \square \mathcal{G}_2)$
Eigenvalue	λ_i	μ_j	$\lambda_i + \mu_j$
Eigenvector	v_i	u_j	$v_i \otimes u_j$

- Also $(v_i \otimes u_i)^T (B(S_1) \otimes B(S_2)) = v_i^T B(S_1) \otimes u_i^T B(S_2)$
- The proof follows from these observations.

Controllability Factorization - Layered Control

Theorem 2: Layered Controllability

The dynamics

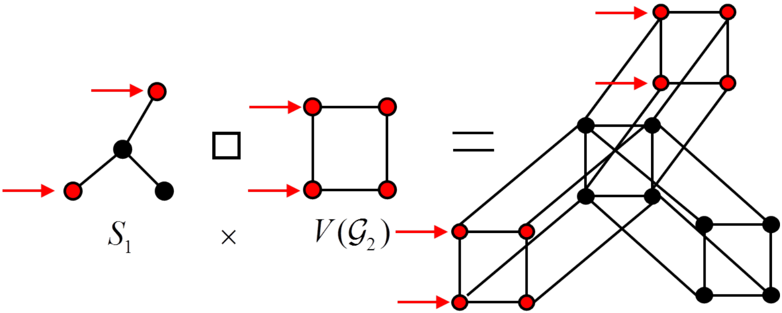
$$\dot{x}(t) = -A(\prod_{\square}\mathcal{G}_i)x(t) + B(\prod_{\times} S_i)u(t)$$
$$y(t) = C(\prod_{\times} R_i)x(t)$$

where $S_i = R_i = V(\mathcal{G}_i)$ for $i = 2, \ldots, n$ is controllable/observable if and only if

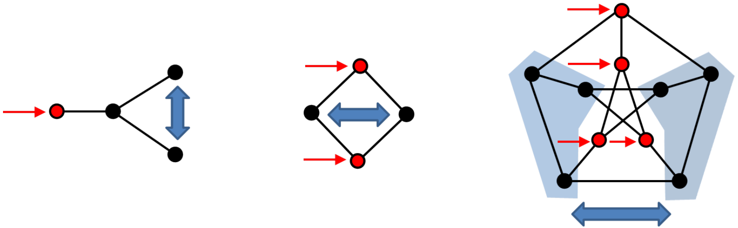
$$\dot{x}_1(t) = -A(\mathcal{G}_1)x_1(t) + B(S_1)u_1(t)$$
$$y_1(t) = C(R_1)x_1(t)$$

is controllable/observable.

Controllability Factorization - Layered Control



Uncontrollability through Symmetry



Proposition (Rahmani and Mesbahi 2006)

$(A(\mathcal{G}), B(S))$ is uncontrollable if there exists an automorphism of $\mathcal{G}$ which fixes all inputs in the set S (i.e., S is not a determining set.)

The determining number of a graph $\mathcal{G}$, denoted $Det(\mathcal{G})$, is the smallest integer r so that $\mathcal{G}$ has a determining set S of size r.

Corollary

$(A(\mathcal{G}), B(S))$ is uncontrollable if $|S| < Det(\mathcal{G})$.

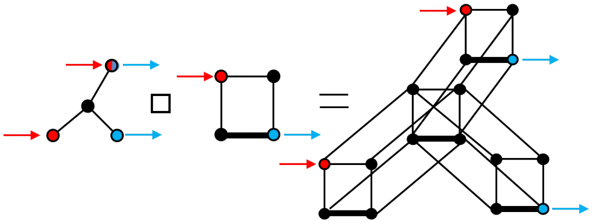
Breaking Symmetry

Automorphism group for graph Cartesian products

The automorphisms for a connected $\mathcal{G}$ is generated by the automorphisms of its prime factors.

Proposition: Automorphism group for graph Cartesian products

For controllable pairs $(A(\mathcal{G}_1), B(S_1))$ and $(A(\mathcal{G}_2), B(S_2))$ where $|S_1| = Det(\mathcal{G}_1)$ and $|S_2| = 1$. Then $S = S_1 \times S_2$ is the smallest input set such that $A(\mathcal{G}_1 \square \mathcal{G}_2, B(S))$ is controllable.



Conclusion

- Explored decomposition of Z-matrix based network into smaller factor-networks
- Provided exact and approximate factorization of state dynamics
- Presented a factorization of controllability - a product and layered approach
- Linked the factors symmetry to smallest controllable input set
- Future work involves examining other graph products in network dynamics