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Time-scale Separation on Networks: Consensus, Tracking, and State-dependent Interactions

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***Abstract*—This paper studies the coupling between dynamics that span multiple time-scales in distributed networked systems. In particular, we consider the evolution of the consensus dynamics interacting with fast nonlinear vehicle dynamics as well as its progress over a state-dependent graphs with slow-varying weight dynamics. Graph-based guarantees are provided that certify the existence of a separation principle across time-scales. Further, we quantify the role of the network's structure in ensuring stability of the composite multi-time-scale system. Graph spectral measures are calculated providing designers a network structure approach to improve performance and/or stability of the coupled system. Examples are presented illustrating the results.**

***Index Terms*—Time-Scale Separation; Networked Dynamic Systems; State-Dependent Networks**

I. Introduction

Consensus-based systems are an effective approach for networked, multi-agent systems in settings such as multi-vehicle control, formation control, swarming, and distributed estimation [1], [2]. In their simplest form, consensus-based systems describe linear diffusive coupling between agents in a static network. In practice, however, the consensus based system is often coupled to nonlinear tracking and dynamic weight evolution that operate over separate time-scales. This dynamic coupling introduces two challenges for analysis. First, even if both sets of dynamics are stable individually, their interconnection is not necessarily stable. Second, the coupling hides and potentially changes the graph properties that underlie the consensus dynamics even if stability is maintained. It is thus unclear whether features such as consensus and the rate of convergence will be invariant when coupled with another dynamic system.

This paper considers two specific cases of network coupling over multiple time-scales. The first case considered is a *consensus tracking problem,* where agents with nonlinear dynamics must "quickly" track a reference given by a slow consensus dynamics. Tracking is an essential assumption in much of the literature on distributed robotics [3], [4], but a reference trajectory may not be immediately realizable by agents that are subject to their own nonlinear dynamics. Further, many designers assume arbitrary levels of tracking performance, which is sometimes referred to as tracking a virtual vehicle or virtual particle [5]. This allows, for example, distributed formation controllers to be formulated that approximately decouple a formation's shape and its center of mass [6]. The second case considered in this paper is a *state-dependent graph problem,* where agents follow a quickly evolving consensus dynamics whose underlying network interactions have slowly varying dynamics. Such scenarios have been investigated from several different directions in the past. In [7] a method was proposed for maximizing the second smallest eigenvalue of the graph Laplacian when edge weights are dependent on inter-agent distances. Controllability was considered in [8] for discrete-event, finite-state distributed systems operating over a graph whose edges are dependent on relative states. On another front, a heterophilious form of opinion dynamics was analyzed in [9] where edges are strengthened between agents with disparate opinion states.

In this work, we apply tools from singular perturbation theory to analyze these multiple time-scale networked systems. In the network systems literature, singular perturbation theory has been primarily applied to explore time-scale separation within the network caused by weak connections. This separation has been exploited to formulate reduced order models for large power networks by area-aggregation [10], dynamic equivalence [11], and slow coherency [12]. Weak inter-node connections have been characterized with respect to the graph structure in [13]. Further, singular perturbation methods have been leveraged to design favorable spectral graph properties in the partial edge design problem [14]. An overview of singular perturbation theory may be found in [15], including its use in control design.

The primary contributions of this work are: 1) A formulation of reduced order models that formalize the applicability of a separation principle between the consensus dynamics and other coupled dynamics at both slower and faster time-scales; 2) The development of quantitative bounds with respect to the underlying graph topology that guarantee asymptotic stability of the composite system.

The structure of the paper is as follows. In §II, we present the background material. In §III and §IV, we investigate time-scale separation for the consensus tracking and state dependent networks, respectively. The main results are illustrated for specific examples in §V, and conclusions are presented in §VI.

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II. Background

In this section, a brief background is provided on the notation, dynamics and definitions used in this paper.

The consensus problem is defined on an undirected, weighted graph $\mathcal{G} = (V, E, W)$ that consists of a node set V with cardinality $|V| = n$, an edge set E with cardinality $|E| = m$, and a positive weight set W with cardinality $|W| = m$ and associated vector of weights w under some ordering [2]. In this context, $x_i(t) \in \mathbb{R}^d$ represents the d states of agent $i \in V$ at time t, and each agent's states evolve directly relative to the state measurements of neighboring agents in the graph $\mathcal{G}$. The dynamics are compactly represented over all agents in the graph as

$$\dot{x}(t) = -(L(\mathcal{G}) \otimes I_d)x(t), \quad (1)$$

where $x(t) = \begin{bmatrix} x_1(t)^T & \cdots & x_n(t)^T \end{bmatrix}^T \in \mathbb{R}^{nd}$ and $L(\mathcal{G})$ is the Laplacian matrix whose eigenvalues $0 = \lambda_1(\mathcal{G}) \leq \cdots \leq \lambda_n(\mathcal{G}) = \lambda_{\max}(\mathcal{G})$ are associated with the normalized eigenvectors $\frac{1}{n}\mathbf{1} = v_1, v_2, \ldots, v_n$. An attraction of these dynamics is that all agents converge to the consensus subspace $\mathcal{X}_c = \{x \in \mathbb{R}^{nd} \,|\, x_1(t) = \cdots = x_n(t)\}$ when $\mathcal{G}$ is a connected graph.

For a small, positive perturbation parameter ε, a system

$$\begin{aligned} \dot{x} &= f(x, z; \varepsilon);\ x(t=0) = x(0) \\ \varepsilon \dot{z} &= g(x, z; \varepsilon);\ z(t=0) = z(0) \end{aligned} \quad (2)$$

(with x not necessarily a set of agent states) is said to be in *standard singularly perturbed form* if the vector fields are bounded and continuously differentiable in their arguments, and if each root $z = h(x)$ of the algebraic equation $0 = g(x, z; 0)$, found by setting $\varepsilon = 0$ in (2), are isolated [16]. The states $x \in D_x \subset \mathbb{R}^{n_s}$ are then called the *slow states* and $z \in D_z \subset \mathbb{R}^{n_f}$ the *fast states*.

Of particular interest are the intuitive notions of the *Reduced Slow System* which describes the dynamics of the slow states $x^{(0)}$ of (2) as if the fast states are always at an isolated equilibrium $h(x)$, and the *Reduced Fast System* which describes the dynamics of the fast states of (2) over a stretched time-scale $\tau = t/\varepsilon$ for which the $x^{(0)}$ are assumed constant. The results in this paper make use of the Tikhonov theorem in the singular perturbation literature, which states the conditions over which these reduced order models are valid (see [16] for details).

III. Consensus Tracking Dynamics - Slow Consensus

A common assumption of controllers designed with the consensus dynamics (1) is that the underlying system has particle dynamics. In many applications, however, this is not the case and as such a particle tracking controller must be designed in conjunction with the consensus dynamics. For fast tracking dynamics relative to the slow consensus dynamics, the coupled consensus-tracking dynamics is singularly perturbed. The following formulates this coupled system in a singular perturbation framework where the tracking dynamics are fast compared to the slow consensus dynamics.

Let the true state of agent i be $z_i(t) = \left[z_{i,1}(t)^T\, z_{i,2}(t)^T\right]^T \in \mathbb{R}^{n_z}$ of which $z_{i,1}(t) \in \mathbb{R}^d$ is the true position of agent i, and let the tracked virtual position of agent i be $x_i(t) \in \mathbb{R}^d$. Further, let the tracking dynamics $g_i(x_i(t), z_i(t))$ be designed such that $z_{i,1}(t) \to x_i(t)$ and $z_{i,2}(t) \to 0$.

Then, the combined dynamics of all agents can be represented in the standard singularly perturbed form as[1]

$$\dot{x} = -(L(\mathcal{G}) \otimes I)z_x \quad \text{(3a)}\ ; \qquad \varepsilon \dot{z} = g(x, z), \quad \text{(3b)}$$

where $g(x, z) = \left[g_1(x_1, z_1)^T \cdots g_n(x_n, z_n)^T\right]^T$, $z = \left[z_1^T \cdots z_n^T\right]^T$, $z_x = \left[z_{1,1}^T \cdots z_{n,1}^T\right]^T$, and ε is a small parameter.

Typical consensus-tracking dynamic formulations advocate a fast tracking dynamics so that, loosely speaking, the dynamics (3) can be approximated by dynamics (1). The following section, formalizes this process and presents graph-based measures for bounds on ε which provide the necessary requirement to assert the separation of the slow consensus dynamics from the fast tracking dynamics. To this end we provide the following assumptions on the tracking function $g_i(x_i, z_i)$ common to all agents:

Assumptions: 1) The function $g_i(x_i, z_i)$ is continuously differentiable with respect to x and z over their respective domains and has a unique, isolated zero, namely $z_i = h_i(x_i) = \left[x_i^T\, \mathbf{0}^T\right]^T$; **2)** Defining the error term $\hat{z}_i := \left[\hat{z}_{i,1}^T\, \hat{z}_{i,2}^T\right]^T$ with $\hat{z}_{i,1} = z_{i,1} - x_{\text{ref},i}$ and $\hat{z}_{i,2} = z_{i,2}$, there exists a positive-definite Lyapunov function $V_{\text{fast},i}(\hat{z}_i)$ for all $z_i \in D_z$ and $x_{\text{ref},i} \in D_x$ such that $\frac{\partial V_{\text{fast},i}}{\partial \hat{z}_i} g_i(x_i, \hat{z}_i + h_i(x_i)) \leq -\alpha_2 \Phi_i^2(\hat{z}_i)$, with $\alpha_2 > 0$ and $\Phi_i^2(\hat{z}_i)$ a continuous scalar function with $\Phi_i(0) = 0$.

Assumption 1 states that the tracking dynamics $\dot{z}_i = g_i(x_{\text{ref},i}, z_i)$ has the unique equilibrium $z_{i,1} = x_{\text{ref},i}$ and $z_{i,2} = 0$. Assumption 2 then states that these dynamics are asymptotically stable uniformly for all fixed $x_{\text{ref},i}$,. This is reasonable, for example, when the agents' domain is relatively homogenous so that a single bound may be found for convergence to a reference point.

A. Separation Principle

The singular perturbation form (3) indicates that the system exhibits multiple time-scale behavior as $\varepsilon \to 0$. In order to specify reduced order models based on this time-scale separation, note that the sole root $h(x)$, or *quasi-steady-state* value, of (3b) occurs for $z = h(x) = \left[h_1(x_1)^T \cdots h_n(x_n)^T\right]^T$ and $0 = g(x, h(x))$. Applying Assumption 1 to (3), the following reduced slow system is defined.

Definition 3.1. The Reduced Slow System of (3) is

$$\dot{x}^{(0)} = -(L(\mathcal{G}) \otimes I)z_x^{(0)} = -(L(\mathcal{G}) \otimes I)x^{(0)} \quad (4)$$

subject to the initial condition $x^{(0)}(0) = x(0)$.

The Reduced Slow System describes the dynamics of the slow states when the fast states have reached equilibrium.

[1]For brevity, the time t indication on the states will be removed henceforth.

For this case, the Reduced Slow System corresponds to the dynamics when the agents are running particle consensus dynamics (1).

For a given starting initial state of the model (3) and $\varepsilon \neq 0$, a natural question is how well does $x^{(0)}$ estimate x, and similarly how close are z_x and x. To aid the evaluation of this question, a new stretched time variable $\tau = t/\varepsilon$ is introduced. In this time-scale, the variables x is slowly varying. Consequently, a reduced fast system dynamics can be formed with respect to a fixed x, as follows:

Definition 3.2. The Reduced Fast System of (3) is

$$\frac{d}{d\tau}\hat{x} = 0 \quad ; \quad \frac{d}{d\tau}\hat{z} = g(\hat{x}, \hat{z} + h(\hat{x})) \tag{5}$$

subject to the initial condition $\hat{x}(\tau = 0) = x^{(0)}$ and $\hat{z}(\tau = 0) = z(0) - h(x^{(0)})$.

The Reduced Fast System describes the evolution of the fast states z evolving towards the equilibrium $z = h(x^{(0)})$ over a stretched time-scale τ whereby the slow state is perceived as fixed.

The machinery is now in place to address the accuracy of proximity of dynamics model (4) to (3). This is formally posed in the following theorem.

Theorem 3.3. *Under the dynamics (3) and with Assumptions 1 and 2, there exists an $\varepsilon_0 > 0$, such that for $0 \leq \varepsilon \leq \varepsilon_0$ the approximations $x(t) = x^{(0)}(t) + \mathcal{O}(\varepsilon)$ defined in (4) and $z(t) = \hat{z}(\tau) + h(x^{(0)}(t)) + \mathcal{O}(\varepsilon)$ defined in (5) are valid for all $t \in [0, T]$, and there exists a $t_1 > 0$ such that the approximation $z(t) = h(x^{(0)}(t)) + \mathcal{O}(\varepsilon)$ is valid for $t \in [t_1, T]$.*

Proof: The result follows from Tikhonov theorem [16], by noting Assumption 1 for the continuous function $g(x, z)$ and the formulation of the problem in standard form. ■

Theorem 3.3 states the conditions under which the reduced models (4) and (5) are good approximations of the system (3). In turn, after some time t_1 the tracking consensus dynamics (3) will be well approximated by just the particle consensus dynamics (1).

B. Bounds of ε and Graph-Based Interpretation

While Theorem 3.3 provides a certificate of the existence of an ε_0 below which the given approximations are valid, a quantitative result is required for implementation purposes. This is because bounds on ε directly affect the viability of the tracking dynamics. Specifically, there are practical limitations on how fast updates of z can be applied.

To this end a quantitative bound on ε can be formed to guarantee convergence of the dynamics (3) to consensus. The result is summarized in the following theorem.

Theorem 3.4. *Under the dynamics (3) and with Assumptions 1 and 2, if there exists "mixing" constants $\beta_1, \beta_3, \gamma_2 \geq 0$ such that*

1) $-x_\perp^T(L(\mathcal{G}) \otimes I)\hat{z}_x \leq \beta_1 \|x_\perp\|_2 \Phi(\hat{z})$

2) $\frac{\partial V_{fast}}{\partial \hat{z}_x}(L(\mathcal{G}) \otimes I)(\hat{z}_x + x_\perp) \leq \beta_3 \|x_\perp\|_2 \Phi(\hat{z}) + \gamma_2 \Phi(\hat{z})^2$

for all $x \in D_x$ and $z \in D_z$, where $x_\perp := x - \frac{1}{n}(\mathbf{1}\mathbf{1}^T \otimes I)x$, then the consensus subspace $\mathcal{X}_c$ is asymptotically stable for all $0 < \varepsilon < \varepsilon^*$, where

$$\varepsilon^* = \alpha_2 / \left(\gamma_2 + \frac{\beta_1\beta_3}{\lambda_2(\mathcal{G})}\right)$$

and α_2 is defined in Assumption 2.

Proof: The result follows by calculating the coefficients outlined as steps (i)-(v) in [16, pp. 35–39] which are used to bound ε^*. Examining the quadratic Lyapunov function $V_{\text{slow}}(x) = \frac{1}{2}x^T x$ then

$$-\frac{\partial V_{\text{slow}}}{\partial x}(I \otimes L(\mathcal{G}))x$$
$$\leq -\lambda_2(\mathcal{G})(x - (\mathbf{1} \otimes I)x_\parallel)^T(x - (\mathbf{1} \otimes I)x_\parallel) = -\alpha_1\Psi^2(x),$$

where $\Psi^2(x) = x_\perp^T x_\perp$, $\alpha_1 = \lambda_2(\mathcal{G})$, and $x_\parallel = \frac{1}{n}(\mathbf{1}^T \otimes I)x$ is the centroid component of each dimension of x; α_2 is provided from Assumption 2, and $\gamma_1 = \beta_2 = 0$ as $g(x, z)$ is independent of ε. With $f(x, z) = -(L \otimes I)z_x$ then

$$\frac{\partial V_{\text{slow}}}{\partial x}[f(x, \hat{z} + h(x)) - f(x, h(x))] = -x^T(L \otimes I)\hat{z}_x$$
$$\leq -x_\perp^T(L \otimes I)\hat{z}_x\beta_1 \|x_\perp\|_2 \Phi(\hat{z}) = \beta_1\Psi(x)\Phi(\hat{z}).$$

Since $\frac{\partial V_{\text{fast}}}{\partial x} = 0$ and $\frac{\partial V_{\text{fast}}}{\partial \hat{z}}\frac{\partial h}{\partial x} = \frac{\partial V_{\text{fast}}}{\partial \hat{z}_x}$ then

$$\left[\frac{\partial V_{\text{fast}}}{\partial x} - \frac{\partial V_{\text{fast}}}{\partial \hat{z}}\frac{\partial h}{\partial x}\right] f(x, \hat{z} + h(x))$$
$$= \frac{\partial V_{\text{fast}}}{\partial \hat{z}_x}(L \otimes I)(\hat{z}_x + x) = \frac{\partial V_{\text{fast}}}{\partial \hat{z}_x}(L \otimes I)(\hat{z}_x + x_\perp)$$
$$\leq \beta_3 \|x_\perp\|_2 \Phi(\hat{z}) + \gamma_2\Phi(\hat{z})^2 = \beta_3\Psi(x)\Phi(\hat{z}) + \gamma_2\Phi(\hat{z})^2.$$

Finally, applying the acquired variables to the proscribed upper bound for ε, the result follows. ■

Theorem 3.4 provides a sufficient bound to guarantee convergence of the consensus tracking dynamics. The bound is improved as the speed of the tracking dynamics is increased, characterized through α_2. The mixing constant β_1 bounds differences in the consensus dynamics between the perfect tracking of the Reduced Slow System (4) and the complete model (3a), with a larger difference yielding a worse ϵ^*. Similarly, the mixing constants β_3 and γ_2 examine the effects of differences between the constant reference of the Reduced Fast System (5) and the complete model (3b) on the consensus dynamics, with a larger effect again yielding a worse bound. Further, the theorem indicates graph-based features which enable larger ε^*, specifically a ratio of $\beta_1\beta_3$ to $\lambda_2(\mathcal{G})$ being close to unity. The graph-based features become more clear when Assumption 2 admits a special Lyapunov function. In this case, Theorem 3.4 can be further refined as follows.

Corollary 3.5. *For the dynamics and assumption of Theorem 3.4 with $V_{fast} = \hat{z}_x^T Q \hat{z}_x + V_{fast,2}(\hat{z}_{2,1}, \ldots, \hat{z}_{2,n})$ for $0 \preceq Q \preceq qI$ and $\Phi(\hat{z})^2 = \hat{z}_x^T \hat{z}_x$ in Assumption 2, then*

$$\varepsilon^* = \alpha_2 / \left(\lambda_{max}(\mathcal{G})q\left(1 + \frac{\lambda_{max}(\mathcal{G})}{\lambda_2(\mathcal{G})}\right)\right).$$

Proof: Theorem 3.4 is satisfied with use of the Cauchy-Schwarz inequality and using $\beta_1 = \lambda_{\max}(\mathcal{G})$ and $\beta_3 = \gamma_2 = \lambda_{\max}(\mathcal{G})q$.

It is now clear that larger ε^* occurs when $\lambda_{\max}(\mathcal{G})$, which is correlated with nodes with a large degree, is small while the ratio $\lambda_2(\mathcal{G})/\lambda_{\max}(\mathcal{G})$ is close to unity. To this effect, regular graphs and expander graphs will be particularly favorable.

IV. State Dependent Graph - Fast Consensus

In many practical systems, the network underlying the consensus protocol may have its own dynamics that interact with the states of the agents. The following formulates this coupled system in a singular perturbation framework where the agent dynamics are fast compared to the slow dynamics of the states underlying the network.

Let the set of agent states be $z(t) \in D_z \subset \mathbb{R}^n$, and the undirected, weighted graph whose weights depend on the slowly-changing variables $x(t) \in D_x \subset \mathbb{R}^m$ be $\mathcal{G}_x = (V, E, W(x))$. Then, the consensus system with a state-dependent network can be written as[2]

$$\dot{x} = f(x, z) \quad \text{(6a)} \; ; \qquad \varepsilon \dot{z} = -L(\mathcal{G}_x) z \quad \text{(6b)}$$

Of particular interest is understanding when the dynamics of the state-dependent network are guaranteed to allow the agents to reach consensus. To this end, the following assumptions are adopted on (6):

Assumptions: 1) The functions $f(x, z)$ and $L(\mathcal{G}_x)$ are continuous and differentiable with respect to x and z over their respective domains; **2)** The graph $\mathcal{G}_x$ starts and remains connected over D_x[3]; **3)** There exists a positive-definite Lyapunov function V_{slow} in the domain $x \in D_x$ about the equilibrium set $x_{\text{eq}} \in D_x$ that satisfies $\frac{\partial V_{\text{slow}}}{\partial x} f\left(x, z_\| \mathbf{1}\right) \leq -\alpha \Psi^2(x, z_\|)$, with $\alpha > 0$, $\Psi\left(x, z_\|\right) \in \mathbb{R}$ a continuous function that satisfies $\Psi\left(x_{\text{eq}}, z_\|\right) = 0$, and $z_\| := \frac{1}{n}\mathbf{1}^T z(0)$.

Assumption 3 states that the network's dynamics are asymptotically stable to some set in the network states x when the agents are fixed at consensus. This is reasonable, for example, if the weights of the network change based on distance between the agents so that the weights are stable when the agents have reached agreement.

The system (6) is not in standard form [16] because the nullspace of $L(\mathcal{G}_x)$ has rank of at least one and thus the roots of $0 = -L(\mathcal{G}_x)z$ are not isolated. Equation (6) is therefore an example of a "singular singular perturbation" problem [17], where a slow state is "hidden" in the nominally fast z dynamics. Standard singularly perturbed form is important because it allows an intuitive reduced outer system to be well-defined. Therefore, in the following the problem will be treated by transformation to a standard singular perturbation problem.

[2]Again for brevity, the time t indication on the states will be removed henceforth.

[3]This can be similarly extended for the case of ℓ components of the graph that remain connected in the domain of interest.

A. Transformation to standard form

Since $L(\mathcal{G}_x)$ is connected and undirected in D_x from Assumption 2, v_1 is the sole eigenvector associated with a zero eigenvalue so that $v_1^T L(\mathcal{G}_x) = \mathbf{0}$ and furthermore v_1 is *not* dependent on x within D_x. Now, define the superstate $z_\|$ as the average of z by $z_\| = v_1^T z$ and represent the remaining $n-1$ states by a transformation with any orthonormal basis C of $\mathbb{R}^n \backslash \mathbf{1}$ as $z_\perp = Cz$ so that $Q = \left[v_1^T ; \, C\right]$ is an orthogonal matrix. Therefore,

$$\begin{aligned} \frac{dz_\|}{dt} &= \frac{\partial z_\|}{\partial z}\frac{dz}{dt} + \frac{\partial z_\|}{\partial x}\frac{dx}{dt} \\ &= -\frac{1}{\varepsilon} v_1^T L(\mathcal{G}_x) z + \left(\frac{\partial v_1}{\partial x}\frac{dx}{dt}\right)^T z = 0, \end{aligned}$$

as $v_1^T L(\mathcal{G}_x) = 0$ and $\frac{\partial v_1}{\partial x} = 0$. Therefore $z_\|$ is a fixed parameter of the system. Similarly, the derivative $\frac{dz_\perp}{dt}$ is expanded as

$$\begin{aligned} \frac{dz_\perp}{dt} &= \frac{\partial z_\perp}{\partial z}\frac{dz}{dt} + \frac{\partial z_\perp}{\partial x}\frac{dx}{dt} \\ &= C\left(-\frac{1}{\varepsilon} L(\mathcal{G}_x) z\right) + \frac{\partial}{\partial x}(Cz)\, f\left(x, z_\perp; z_\|\right) \\ &= -\frac{1}{\varepsilon} C L(\mathcal{G}_x) C^T z_\perp \equiv \frac{1}{\varepsilon} g_{\text{fast}}(x, z_\perp; z_\|), \end{aligned}$$

where $\frac{\partial (Cz)}{\partial x} = 0$. . The original dynamics (6) written under this transformation becomes

$$\dot{x} = f\left(x, z_\perp; z_\|\right); \; \varepsilon \dot{z}_\perp = g_{\text{fast}}(x, z_\perp; z_\|), \qquad (7)$$

under the new associated domain $z_\perp \in D_{z_\perp} \subset \mathbb{R}^{n-1}$. In this representation, the "hidden" slow state $z_\|$ has been separated from the nominally fast dynamics and revealed as a parameter dependent only on the initial conditions. The resulting system (7) is therefore the in standard form with slow states x and fast states $z_\perp$.

B. Separation Principle

As $\varepsilon \to 0$, the system (7) exhibits multiple time-scale behavior which indicates that reduced-order models may be applicable. To this end, define the following reduced models of the transformed system:

Definition 4.1. The Reduced Slow System of (7) is

$$\dot{x}^{(0)} = f\left(x^{(0)}, \mathbf{0}; z_\|\right); \; x^{(0)}(0) = x(0) \qquad (8)$$

Definition 4.2. The Reduced Fast System of (7) is

$$\frac{d}{d\tau}\hat{z}_\perp = -C L(\mathcal{G}_x) C^T \hat{z}_\perp; \; \hat{z}_\perp(0) = z_\perp(0). \qquad (9)$$

where x is considered a fixed parameter.

The Reduced Slow System describes the x dynamics when the agents are always at consensus, while the Reduced Fast System describes the evolution of the agents towards consensus over a stretched time-scale when the graph weights are fixed. In this context, Assumption 3 implies that the reduced slow system is asymptotically stable over the domain. With these definitions in place, the following theorem certifies the applicability of using the reduced models (8) and (9).

Theorem 4.3. *Under the dynamics (7) and with Assumptions 1 and 2, there exists an* $\varepsilon_0 > 0$*, such that for* $0 \leq \varepsilon \leq \varepsilon_0$ *the approximations* $x(t) = x^{(0)}(t) + \mathcal{O}(\varepsilon)$ *defined in (8) and* $z_\perp(t) = \hat{z}_\perp(\tau) + \mathcal{O}(\varepsilon)$ *defined in (9) are valid for all* $t \in [0, T]$*, and there exists a* $t_1 > 0$ *such that the approximation* $\hat{z}_\perp = \mathbf{0} + \mathcal{O}(\varepsilon)$ *is valid for* $t \in [t_1, T]$.

Proof: The result follows from the Tikhonov theorem [16] by noting Assumption 1, the formulation of the problem into standard form, and that construction of g_{fast} along with Assumption 2 yields the origin as the globally, asymptotically stable equilibrium of the Reduced Fast System (9). ∎

Theorem 4.3 states the conditions under which the reduced models (8) and (9) are good approximations of the transformed system (7), and thus descriptive of the original system (6). Further, it states that there is a time t_1 after which the dynamics (7) are well approximated by the reduced slow dynamics (8) where the agents have reached consensus at their initial average.

C. Bounds on ε and Allowable Domains

In the following, the stability of the complete, full-order system (7) is analyzed in terms of the properties of the reduced order models (8) and (9). This yields quantitative information, as measured by a lower bound on the perturbation parameter ε, on how the stability properties and domain of convergence of the full-order system depends on the time-scale separation between the slow dynamics of the network parameters and the fast dynamics of the agents seeking consensus. The results are summarized by the following theorem.

Theorem 4.4. *Under the dynamics (6) and with Assumptions 1, 2, and 3, if there exists "mixing" constants* $\beta, \gamma > 0$ *such that*

1) $\frac{\partial V_{slow}}{\partial x}\{f(x, z_\perp; z_\parallel) - f(x, \mathbf{0}; z_\parallel)\} \leq \beta \Psi(x, z_\parallel) \|z_\perp\|_2$
2) $\mathbf{1}^T \frac{\partial w}{\partial x} f(x, z_\perp; z_\parallel) \leq \gamma$

for all $x \in D_x$ *and* $z_\perp \in D_{z_\perp}$*, then the set* $(x, z) = (x_{eq}, z_\parallel \mathbf{1})$ *is asymptotically stable for all* $0 < \varepsilon < \varepsilon^*$*, where*

$$\varepsilon^* = \left(\min_{x \in D_x} \lambda_2(\mathcal{G}_x) \right)^2 / \gamma.$$

Proof: The proof follows the weighted Lyapunov function approach outlined as steps (i)-(v) in [16, pp. 35–39]. A composite Lyapunov function $V = (1-d) V_{\text{slow}} + d V_{\text{fast}}$, with $0 < d < 1$ is constructed from the Reduced Slow System Lyapunov function V_{slow} and a Reduced Fast System Lyapunov function V_{fast}. The Lyapunov function V_{fast} is constructed for the Reduced Fast System (9) as $V_{\text{fast}} = \frac{1}{2}\lambda_2(\mathcal{G}_x) z_\perp^T z_\perp$ which gives $-\frac{\partial V_{\text{fast}}}{\partial z_\perp} C L(\mathcal{G}_x) C^T z_\perp \leq -\left(\min_{x \in D_x} \lambda_2(\mathcal{G}_x)\right)^2 \|z_\perp\|_2^2$ due to the properties of the Laplacian and construction of the fast state $z_\perp$, and $\partial V_{\text{fast}}/\partial x = \frac{1}{2}\left(\partial \lambda_2(\mathcal{G}_x)/\partial w\right)\left(\partial w/\partial x\right) z_\perp^T z_\perp$ so that $\frac{\partial V_{\text{fast}}}{\partial x} f(x, z_\perp; z_\parallel) \leq \mathbf{1}^T \left(\partial w/\partial x\right) f(x, z_\perp; z_\parallel) \|z_\perp\|_2^2$ using $(\partial \lambda_2/\partial w_{ij}) = (v_i - v_j)^2 \leq 2$ [18]. Further note that g_{fast} is not dependent on ε. The maximum ε is then found (along with the associated d) that guarantees $\dot{V} < 0$. ∎

Theorem 4.4 provides a sufficient bound for convergence to consensus of the agents at their original average, given that the slow dynamics of the network is separately asymptotically stable. The mixing constant β bounds the difference in the graph dynamics between the instantaneous-consensus dynamics of the Reduced Slow System (8) and the complete model (6a), and by construction of the proof a larger difference will negatively affect the domain of attraction. The mixing constant γ examines the effects of differences between the constant-graph dynamics of the Reduced Fast System (9) and the complete model (6b) on the graph dynamics, with a larger difference yielding a worse ε^*. Further, the ε^* bound directly varies with the worst case (least stable) $\lambda_2(\mathcal{G}_x)$ in the domain.

V. Examples

A. Unicycle Tracking Particle in Consensus Network

Consider the particle-consensus reference dynamics

$$\dot{x} = -(L(\mathcal{G}) \otimes I) z_x \tag{10}$$

where $x \in \mathbb{R}^n$ is the desired distance of agents from the origin, $z_x \in \mathbb{R}^n$ is the true distance, and the underlying agents are unicycle-type robots. That is, the ith agent is governed by unicycle dynamics where $p_i \in \mathbb{R}^2$ is the agents position and θ_i its bearing. This section proposes a tracking controller for the unicycle dynamics, analyzes its convergence, and examines the standard singular perturbation model induced by fast tracking of the reference.

Adapting the controller proposed in [19] to track the desired distance x_i from the origin with associated bearing pointing at the origin, the closed loop controller is

$$\dot{z}_{i,1} = -k_d \cos^2 z_{i,2}(z_{i,1} - x_i); \;\; \dot{z}_{i,2} = -k_\alpha z_{i,2}, \tag{11}$$

where $k_d, k_\alpha > 0$, $z_{i,1} = \|p_i\|$ is the distance to the origin, and $z_{i,2} = \text{atan2}(p_i) + \pi - \theta_i$ the angle between the vehicle principle axis and the distance vector $z_{i,1}$. Adapting the proof by Aicardi *et al.*, the dynamics are asymptotically stable, with $z_{i,1} \to x_i$, and $z_{i,2} \to 0$ shown using the Lyapunov function and its derivative $V_{fast,i} = \frac{1}{2}(z_{i,1} - x_i)^2 + \frac{1}{2} z_{i,2}^2$ and $\dot{V}_{fast,i} = -k_d \cos^2 z_{i,2}(z_{i,1} - x_i)^2 - k_\alpha z_{i,2}^2 \leq -k_d(z_{i,1} - x_i)^2 - \left(k_\alpha - k_d(z_{i,1} - x_i)^2\right) z_{i,2}^2$, respectively.

The dynamics satsifies all the assumptions in Section III. In particular, with $k_\alpha > k_d z_{i,1}^2$ for all $z \in \mathcal{D}_z$, then V_{fast} and $\dot{V}$ are of the form of Corollary 3.5 with $\alpha_2 = k_d$ and $Q = \frac{1}{2} I$. Therefore, under the closed-loop tracking dynamics (10)-(11) the consensus subspace $\mathcal{X}_c$ is asymptotically stable for all $0 < \varepsilon < \varepsilon^*$ where $\varepsilon^* = 2k_d / [\lambda_{\max}(\mathcal{G})(1 + \lambda_{\max}(\mathcal{G})/\lambda_2(\mathcal{G}))]$, by Corollary 3.5.

The dynamics is evaluated on an 8-node cycle graph $\mathcal{G}_C$ and a barbell graph $\mathcal{G}_B$, depicted in Figure 1a, with corresponding ε^* values for $k_\alpha = k_d = 1$ of 0.063 and 0.021, respectively. For both graphs with $\varepsilon = 0.01$, the true distance z_x and desired distance x converge to consensus and one another, with the trajectory of z_x for $\mathcal{G}_C$ displayed in Figure 1b. In Figure 1c, the trajectory of x matches the reduced slow dynamics $x^{(0)}$ for $\varepsilon = 0.01$ for both graphs while in Figure

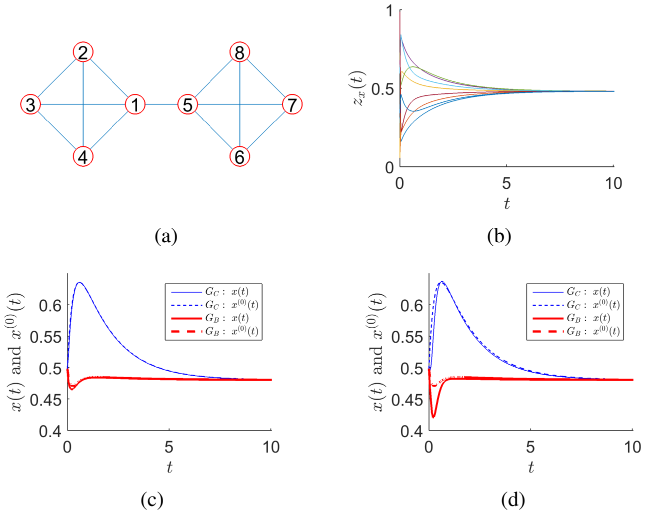

Figure 1: (a) Barbell graph (b) Trajectory of z_x over $\mathcal{G}_C$ for $\varepsilon = 0.01$ (c) Sample trajectory of x and $x^{(0)}$ for $\mathcal{G}_C$ and $\mathcal{G}_B$ with $\varepsilon = 0.01$. (d) Sample trajectory of x and $x^{(0)}$ for $\mathcal{G}_C$ and $\mathcal{G}_B$ for $\varepsilon = 0.06$.

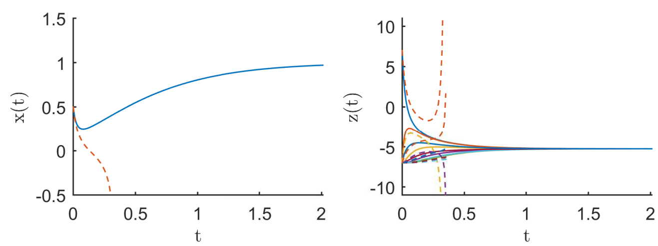

Figure 2: Trajectory of $x(t)$ and $z(t)$ for $\varepsilon = 0.02$ (solid) and $\varepsilon = 0.03$ (dashed).

1d only $\mathcal{G}_C$ performs well for $\varepsilon = 0.06$. This correlates well with the graph-based bounds for ε^* with $\mathcal{G}_C$ with small $\lambda_{\text{max}}(\mathcal{G}_C)$ and $\lambda_{\text{max}}(\mathcal{G}_C)/\lambda_2(\mathcal{G}_C)$, outperforming the dynamics over $\mathcal{G}_B$ with relatively large $\lambda_{\text{max}}(\mathcal{G}_B)$ and $\lambda_{\text{max}}(\mathcal{G}_B)/\lambda_2(\mathcal{G}_B)$.

B. Effect of Weight Profile for a Dynamic Graph

Consider the dynamics (6) defined over $x \in (0.5, \infty)$, $z \in \mathbb{R}^n$ with the slow graph dynamics defined by $f(x, z) = 1 - x^2 - |z_1 - z_2|$ and where the graph $\mathcal{G}_x = (V, E, W(x))$ has state-dependent weights defined by the function $w_{12}(x) = x/n$ and $w_{ij} = 1/n$, otherwise. To analyze this system, a Reduced Slow System Lyapunov function $V_{\text{slow}} = \frac{1}{4}(1 - x^2)^2$ may be defined. Theorem 4.4 is then satisfied with $\gamma = 1/n$ around the equilibrium point $(x, z) = \big(1,\, z_{\|}\mathbf{1}\big)$ with $\varepsilon^* = \min_{x \in D_x} \{\lambda_2(\mathcal{G}_x)\}^2 n$. The minimum $\lambda_2(\mathcal{G}_x)$ occurs when $w_{12} = 0.5/n$. Applying the dynamics on an 8-node cycle graph then $\varepsilon^* = 0.027$. For $\varepsilon = 0.02$, x and z converge to $(1, z_{\|}\mathbf{1})$ supporting Theorem 4.4 while for $\varepsilon = 0.03$ the dynamics diverge indicating the potential for unstable coupled dynamics. The trajectory for x and z for each of these case are displayed in Figure 2.

VI. Conclusion

This paper examines the interactions between multi-time-scale networked dynamic systems and provides guarantees on the stability of these coupled systems. Two scenarios were explored: a slow network/fast tracking problem and a fast network/slow edge weight variation problem. For these scenarios we established perturbation parameter bounds to guarantee stability of the composite system. We drew tools from singular perturbation theory to show our results. Future work of particular interest will extend these results to directed networks as well as explore systems where state-dependent graphs appear in both the slow and the fast dynamics layers.

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