2015 IEEE 54th Annual Conference on Decision and Control (CDC)
December 15-18, 2015. Osaka, Japan

State Controllability, Output Controllability and Stabilizability of Networks: A Symmetry Perspective

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***Abstract*—This paper explores the role of symmetry when establishing output controllability and stabilizability of LTI networked systems. Previous work, which examined graph symmetry through its signed fractional graph automorphisms, has been extended to output controllability and stabilizability. Conditions on input symmetry, output symmetry, and non-output symmetry over the network are proved to provide necessary and sufficient conditions for output controllability. Results pertaining to output stabilizability provide insight into stabilization of negatively weighted consensus dynamics. The approach also provides a semi-definite programming formulation of output controllability and stabilizability.**

***Index Terms*—Network controllability; Network observability; Graph symmetry; Signed fractional automorphisms**

1. Introduction

Networked dynamic systems are increasingly becoming an integral part of our technological world, from robotic networks, to smart grids, to sensor networks. Such systems are also core to understanding complex dynamical behavior in natural systems, including biology, quantum networks, and social systems. As a result, an explosion of research in the area of network systems has resulted [1], [2], [3]. Network controllability and observability stems for the desire to put a control theoretic twist on networked systems. This is due to the observation that once the dynamics of the network has been understood, and potentially examined in the light of the underlying network, it is desired to control it from certain boundary nodes. Such a point of view has proven to be of great importance in the situations where the network behavior can be influenced from certain “ports”. Examples of such scenarios include networked robotic systems, human-swarm interaction, and network security [3], [4], and in areas such as quantum networks with respect to controlled potentials [5].

In the previous works on network controllability, it has been shown that the symmetry of the network, parametrized in terms of the automorphism group of the graph, plays a significant role. These works include networks where the underlying dynamics is driven by the adjacency matrix [5] or the Laplacian matrix [6]. Extensions of the symmetry point of view on network controllability have been extended to nonlinear systems [7]. Furthermore, such investigations have also examined different classes of graphs such as circulants [8], grids [9], random [10], distance regular [11] and Cartesian product graphs [12].

The research of the authors was supported by the U.S. Army Research Laboratory and the U.S. Army Research Office under contract number W911NF-13-1-0340. The authors are with the Department of Aeronautics and Astronautics, University of Washington, WA 98105. Emails: {airliec, mesbahi}@uw.edu.

Existing properties that relate the network's symmetry structure and points where control can be injected into the network provide sufficient conditions for uncontrollability. These conditions can often be reformulated as cardinality requirements on the number of inputs into the graph. We have formalized this concept using the notion of the determining number of the graph [12] - the minimum number of nodes that must be fixed to break all graph symmetries.

Furthermore, it has been shown that these symmetry properties are not necessary for uncontrollability. In our previous work [13], we have shown that the symmetries that exactly correspond to controllability are in fact characterized in terms of the signed fractional automorphisms. This class of automorphisms have been introduced and explored in detail by Scheinerman and Ullman [14]. correspondence between fractional symmetries in the network and the system theoretic properties of the resulting system, in turn, provides a graph-based convex optimization formulation of the controllability and stabilizability problems.

Our contribution in this paper is three folds. First, we show how the symmetry perspective on system theoretic properties of networks can be extended to network outputs. This is particularly relevant in scenarios where it is not required to control the state of every node in the network, rather only control of a linear function of the states is relevant. Our results in this direction point to an intricate alignment/misalignment between symmetries at the input, the output, and internal nodes. Second, we show how our work on network controllability extends to network stabilizability. This work, on the other hand, is of importance when the interactions amongst the nodes in the network necessitate stabilization for desired network-level behavior. And finally, we show how our characterization of network output controllability and stabilizability facilitates a convex optimization approach to efficiently calculate network control.

The organization of the paper is as follows. In Section 2, we begin by introducing relevant background material pertaining to graphs. We proceed in Section 3 to describe the controlled consensus problem. Section 4 contains the core results of the paper providing symmetry based necessary and sufficient conditions for (output) controllability and stabilizability. Concluding remarks are provided in Section 5.

2. Notation and Background

We provide the notation and a brief background on the constructs and models that will be used in this paper. For a column vector $v \in \mathbb{R}^p$, both v_i and $[v]_i$ denote its ith element.

For matrix $M \in \mathbb{R}^{p \times q}$, $[M]_{ij}$ denotes the element in its ith row and jth column. The complex conjugation of matrix $M \in \mathbb{C}^{p \times q}$ is denoted as M^* , its element-wise complex conjugate is $\bar{M}$, its real component by $\text{Re}(M)$ and if M is square its trace by $\textbf{tr}(M)$. The identity matrix is denoted I and e_i is the column vector with all zero entries except $[e_i]_i = 1$. The column vector of all ones is denoted as $\mathbf{1}$. The cardinality of a set S is denoted as $|S|$. For set $T \subseteq S$, the notation $S \setminus T$ describes the set of all elements of S not in T.

A. Graphs

A weighted undirected graph $\mathcal{G} = (V, E, W)$ is characterized by a node set V with cardinality n, an edge set E comprised of unordered pairs of nodes with cardinality m and a weight set W of cardinality m. An edge exists between two neighboring nodes i and j if $\{i, j\} \in E$ with weight $w_{ij} \in W$. Unlike traditional graph models, we also allow the weights w_{ij} to be non-positive. The adjacency matrix of $\mathcal{G}$, denoted $\mathcal{A}(\mathcal{G})$, is an $n \times n$ matrix with $[\mathcal{A}(\mathcal{G})]_{ij} = [\mathcal{A}(\mathcal{G})]_{ji} = w_{ij}$ when $\{i, j\} \in E$ and 0 otherwise. The degree matrix $\Delta(\mathcal{G}) \in \mathbb{R}^{n \times n}$ is a diagonal matrix with $[\Delta(\mathcal{G})]_{ii} = \sum_{\{i,j\} \in E} w_{ij}$. The Laplacian matrix $L(\mathcal{G}) = \Delta(\mathcal{G}) - \mathcal{A}(\mathcal{G})$, with $\mathbf{1}^T L(\mathcal{G}) = \mathbf{0}^T$, $L(\mathcal{G})\mathbf{1} = \mathbf{0}$, eigenvalues $0 = \lambda_1(\mathcal{G}) \leq \lambda_2(\mathcal{G}) \leq \cdots \leq \lambda_n(\mathcal{G})$ and associated normalized eigenvectors $v_1, v_2, \ldots, v_n$. The distance between nodes i and j in a graph, denoted as $d(i, j)$, is the length of the shortest path consisting of edges between nodes i and j.

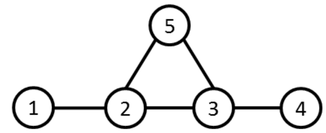
A weighted directed graph $\mathcal{D} = (V, E, W)$ is described by a node set V, an edge set E composed of *ordered* pairs of nodes and weight set W. A directed edge exists from node i to j if $(i, j) \in E$. The directed adjacency matrix is defined such that $[\mathcal{A}(\mathcal{D})]_{ji} = w_{ji} \in W$ when $(i, j) \in E$ and 0 otherwise. The in-degree matrix $\Delta(\mathcal{D})$ is a diagonal matrix with $[\Delta(\mathcal{D})]_{ii} = \sum_{(j,i) \in E} w_{ij}$ and can be used to form the in-degree Laplacian matrix $L(\mathcal{D}) = \Delta(\mathcal{D}) - \mathcal{A}(\mathcal{D})$.

The unweighted undirected graph $\mathcal{G} = (V, E)$ is equivalent to a weighted graph with $w_{ij} = 1$ for all $\{i, j\} \in E$. The unweighted directed graph $\mathcal{D} = (V, E)$ can be similarly defined. Unless stated otherwise, $\mathcal{G}$ and $\mathcal{D}$ refer to unweighted graphs and digraphs, respectively.

B. Signed Fractional Automorphisms

The automorphisms of a graph describe its symmetries and have been previously shown to play an important role in the controllability of dynamics similar to (3.3) [6]. A graph automorphism is a permutation σ of the node set such that $\mathcal{G}$ contains an edge $\{i, j\}$ if and only if it also contains an edge $\{\sigma(i), \sigma(j)\}$. A graph with only a trivial automorphism group is referred to as asymmetric. Every graph automorphism can be represented uniquely as a permutation matrix P which commutes with the adjacency and Laplacian matrix, i.e., $P\mathcal{A}(\mathcal{G}) = \mathcal{A}(\mathcal{G})P$ and $PL(\mathcal{G}) = L(\mathcal{G})P$.

We have previously examined control-theoretic properties of the relaxation of automorphisms to fractional automorphism [13] by examining the following algebraic condition for P to represent an automorphism of the graph $\mathcal{G}$



$$P = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

(a) (b)

Figure 1: (a) Graph $\mathcal{G}$ and (b) matrix P that represents a fractional automorphism on $A(\mathcal{G})$.

$$\mathcal{A}(\mathcal{G})P = P\mathcal{A}(\mathcal{G}),\ \mathbf{1}^T P = \mathbf{1}^T,\ P\mathbf{1} = \mathbf{1}, \tag{2.1}$$

$$P_{ij} \in \{0, 1\}. \tag{2.2}$$

By replacing condition (2.2) with $P_{ij} \in [0, 1]$, creating a doubly stochastic matrix then Scheinerman and Ullman [14] refer to P as describing a *fractional automorphism*. Similarly, by further relaxing the selection of P by removing the condition (2.2), we refer to P as a *signed fractional automorphism*. An example of a fractional automorphism is presented in Figure 1.

It should be noted that unlike permutation matrices, the fact that (signed) fractional automorphisms representing P commutes with the adjacency matrix does not imply it also commutes with $L(\mathcal{G})$. To this end, the (signed) fractional automorphisms are defined explicitly against a specific matrix realization of $\mathcal{G}$. For example, if (2.1) is replaced with $PL(\mathcal{G}) = L(\mathcal{G})P$, then P represents a (signed) fractional automorphism with respect to $L(\mathcal{G})$. Fractional automorphisms also have a relaxed perfect matching interpretation explored by Scheinerman and Ullman [14].

3. Problem Setup

The focus of this paper is the controlled consensus problem [3] defined on a graph $\mathcal{G} = (V, E, W)$, where $x_i(t) \in \mathbb{R}$ represents the state of node $i \in V$ at time t. Interaction between nodes are defined over the edges of the graph E. An external control vector $u(t) \in \mathbb{R}^q$ at time t is applied to node i through some vector $b_i \in \mathbb{R}^q$. An individual node's dynamics is given by $\dot{x}_i(t) = -\sum_{\{i,j\} \in E} w_{ij}(x_i - x_j) + b_i^T u(t)$. In addition, the dynamics at time t is observed by a vector $y(t) \in \mathbb{R}^p$ through a matrix $C \in \mathbb{R}^{p \times n}$. The full system dynamics is then

$$\begin{aligned} \dot{x}(t) &= -L(\mathcal{G})x(t) + Bu(t) \\ y(t) &= Cx(t), \end{aligned} \tag{3.3}$$

where $B = [b_1, b_2, \ldots, b_n]^T \in \mathbb{R}^{n \times q}$. The directed graph version if defined similarly with $L(\mathcal{G})$ replaced with $L(\mathcal{D})$.

It is often of interest where the inputs and outputs of system (3.3) are in terms of the nodes of the graph $\mathcal{G}$. If the set of input nodes in the n node graph is $S = \{i_1, i_2, \ldots, i_q\}$ for $i_1 < i_2 < \cdots < i_q$, the corresponding input matrix is $B = \left[e_{i_1}, e_{i_2}, \ldots, e_{i_q}\right] \in \mathbb{R}^{n \times q}$. We uniquely denote the input matrices of this form as $B_n(S)$. Similarly, the output matrices are defined with respect to an output node set R

as $C_n(R) := B_n(R)^T$. If it is clear from the context, we remove the subscript n for brevity.

As mentioned in Section 2, the weights on the edges of the Laplacian are permitted to be non-positive. As such, the matrix $L(\mathcal{G})$ is not guaranteed to be marginally stable, and for some weight selection, induces clustering. This phenomenon and its connection to effective resistance has been recently examined by Zelazo and Burger [15]. This positive weight relaxation is addressed in this work by examining the stabilizability problem for unstable controlled consensus dynamics (3.3).

The dynamics (3.3) are referred to as controllable (or stabilizable) if the pair (A, B) is controllable (or stabilizable).[1] The dynamics (3.3) are output controllable (or output stabilizable) if the triple (A, B, C) is output controllable (or output stabilizable).

4. Output Controllability and Stabilizability

This section examines conditions on (output) controllability and stabilizability of the general linear dynamics

$$\dot{x} = Ax + Bu, \;\; y = Cx \tag{4.4}$$

as well as specific dynamics (3.3). The machinery required to explore these conditions is the PBH test which provides eigenvector based conditions for (output) controllability and stabilizability. The PBH test is stated in the following Proposition.

Proposition 1. *Popov-Belevitch-Hautus (PBH) test [16]. For $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$. Consider one or more of the following conditions on $\lambda \in \mathbb{C}$ and $v \in \mathbb{C}^n$:*

(a) $v^T B = 0$,

(b) $\exists q \in \mathbb{R}^p$ such that $q^T C = v^T$ and

(c) $Re(\lambda) \geq 0$.

Then there exists a left eigenvalue-eigenvector pair (λ, v) of A such that

1. (a) $\iff$ (A, B) is uncontrollable.

2. (a) and (b) $\iff$ (A, B, C) is output uncontrollable.

3. (a) and (c) $\iff$ (A, B) is unstabilizable.

4. (a), (b) and (c) $\iff$ (A, B, C) is output unstabilizable.

Following the spirit of the initial symmetry and controllability derivation by Rahmani *et al.* [6]. The following theorem presents the (output) controllability and stabilizability conditions with respect to a certificate matrix $P \in \mathbb{R}^{n\times n}$. For clarity of presentation, the proof of Theorem 2 appears in the Appendix.

Theorem 2. *For A diagonalizable and C full row rank, consider one or more of the following conditions on a $P \in \mathbb{R}^{n\times n}$:*

(a) $P \neq I$, $AP = PA$ and $PB = B$.

(b) $P = I - ZC$, for some $Z \in \mathbb{R}^{n\times p}$.

(c) $\frac{1}{2}(P + P^T) \preceq I$ and $PA + (PA)^T \preceq A + A^T$.

Then there exists a $P \in \mathbb{C}^{n\times n}$ such that

1. (a) $\iff$ (A, B) is uncontrollable.

2. (a) and (b) $\iff$ (A, B, C) is output uncontrollable.

3. (a) and (c) $\implies$ (A, B) is unstabilizable.

4. (a), (b) and (c) $\implies$ (A, B, C) is output unstabilizable.

Further, for symmetric A, the statements 3. and 4. are necessary and sufficient.

An attraction of Theorem 2 is that, neglecting the constraint $P \neq I$, the conditions (a), (b) and (c) individually form convex constraints on P. From the proof of Theorem 2, if a $P \neq I$ exists then there exists a $P = I - \frac{1}{2}(wv^* + \bar{w}v^T)$ with $v^* w = 1$ and hence $\mathbf{tr}P = n - 1$ and so an objective that minimizes trace will find a non-identity P if one exists. Consequently, the search for a non-identity P that satisfies any combination of conditions (a), (b) and (c) can be formulated as a convex optimization problem. The output uncontrollability problem corresponds to a non-identity solution of the following optimization problem.[2]

$$\min_{P\in\mathbb{R}^{n\times n}, Z\in\mathbb{R}^{n\times p}} \mathbf{tr}(P) \tag{4.5}$$

$$\text{s.t. } AP = PA, \;\; PB = B$$

$$P = I - ZC. \tag{4.6}$$

For A symmetric, the addition of the constraints

$$\frac{1}{2}(P + P^T) \preceq I, \;\; PA + (PA)^T \preceq A + A^T, \tag{4.7}$$

solves the output unstabilizability condition for the triple (A, B, C). By removing the final conditions (4.6) in (4.5), a convex optimization formulation for controllability can also be presented for general diagonalizable A. The addition of constraints (4.5) solves the stabilizability problem for symmetric A. An attraction of the convex formulation is that additional graph design requirements may be added to the optimization formulation producing a controllability based network synthesis problem. This is left as future work.

The following example exercises the optimization formulation on an output controllability and stabilizability problem.

Example 3. Consider the (output) controllability and stabilizability of the triple (A, B, C_1) and (A, B, C_2) where

$$B = e_2,\; C_1 = e_1^T,\; C_2 = [e_2,\; e_3]^T,\; A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 3 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Noting that A is diagonalizable and C_1 and C_2 are full row rank Theorem 2 is applicable. Applying the pair (A, B) to problem (4.5) without constraint (4.6) gives

$$P = P_1 = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

and so the pair (A, B) is uncontrollable. Solving the optimization problem (4.5) for the triple (A, B, C_1) then $P = I$ and so the dynamics is output controllable. For the triple

[1] Due to the duality between controllability and observability, results pertaining to controllability of the pair (A, B) are equally applicable to observability of the pair (A, B^T).

[2] The optimization problem as stated can produce unbounded solutions for P. As this is an existence-style problem this is not a concern as only the existence of a non-identity solution is of interest. An added constraint of $\mathbf{tr}(P) \geq 0$ can be added without loss of generality if a finite P is required.

(A, B, C_2) then the optimization solution for (4.5) is $P = P_1$ with $Z = \begin{bmatrix} 0 & 0 & 0 \\ -4 & 1 & 1 \end{bmatrix}^T$, and so the dynamics is output uncontrollable. For the triple (A, B, C_2) when the constraints (4.7) are added to the optimization problem (4.5) then the solution is $P = I$ and so output stabilizability is unknown. In fact, from Proposition 1 as A has all positive eigenvalues and as (A, B, C_2) is output controllable then the triple is output unstabilizable, providing an example where a non-symmetric A fails to be necessary and sufficient for statement 4. in Theorem 2.

A. Controlling Digraphs

Generally, most controllability analysis for networked dynamics has been carried out over undirected graphs $\mathcal{G}$. Theorem 2 provides methods for controllability and output controllability on dynamics (3.3) when $L(\mathcal{D})$ is diagonalizable.

To this end, it is worthwhile mentioning classes of digraphs which satisfy the diagonalizability condition.

Acyclic Digraphs: For acyclic digraph $\mathcal{D} = (V, E)$ there exists a node labeling such that $(i, j) \in E$ implies that $i > j$. These are graphs that, if connected, have exactly one rooted spanning tree. Here, $L(\mathcal{D})$ is upper triangular and so diagonalizable.

Simple Laplacian Digraphs: These are digraphs whereby $L(\mathcal{D})$ has no repeated eigenvalues. The fact that all eigenvalues are simple implies $L(\mathcal{D})$ is diagonalizable [17]. A family of directed digraph tournaments with all simple eigenvalue is studied by Caen *et al.* [18].

Normal Laplacian Digraphs: These are digraphs such that $L(\mathcal{D})L(\mathcal{D})^T = L(\mathcal{D})^T L(\mathcal{D})$. The digraph $\mathcal{D}$ must be balanced and eulerian to be a normal Laplacian digraph, i.e., $\sum_{(j,i)\in E} w_{ij} = \sum_{(i,j)\in E} w_{ji}$ for all $i \in V$ and $\mathcal{D}$ contains a cycle that uses every edge exactly once [19]. An attraction of this class of digraphs is that it is easy to check for a digraph's membership algebraically. Further, if $\mathcal{A}(\mathcal{D})$ is normal and $\mathcal{D}$ is regular, i.e., $\Delta(\mathcal{D}) = kI$ for some $k \geq 0$, then $L(\mathcal{D}) = kI - \mathcal{A}(\mathcal{D})$ is also normal as $(kI - \mathcal{A}(\mathcal{D})(kI - \mathcal{A}(\mathcal{D}))^T = (kI - \mathcal{A}(\mathcal{D}))^T(kI - \mathcal{A}(\mathcal{D}))$. Cayley digraphs with normal adjacency matrices, recently studied by Lyubshin and Savchenko [20] are therefore diagonalizable.

Strongly Regular Digraphs: These are regular digraphs such that $A(\mathcal{D})^2 = tI + \lambda A(\mathcal{D}) + \mu\left(\mathbf{1}\mathbf{1}^T - I - A(\mathcal{D})\right)$, for positive integer t, λ and μ. A graph based definition is that regular graph $\mathcal{D}$ is strongly regular if the number of paths of length 2 starting on node i and finishing on node j is t if $i = j$, λ if $(i, j) \in E$ and μ otherwise. The eigenvectors of $L(\mathcal{D})$ have been computed in closed form confirming the matrices diagonalizability [19].

From Theorem 2 and its applicability to strongly regular digraphs we have the following proposition.

Proposition 4. *Let $\mathcal{D}$ be a strongly regular digraph and $B = \alpha\mathbf{1}$ for some $\alpha \in \mathbb{R}$ and $\mathbf{span}(C^T) \supseteq \mathbf{1}^\perp = \{x | \mathbf{1}^T x = 0\}$. Then the triple $(-L(\mathcal{D}), B, C)$ is output uncontrollable.*

Proof: Consider $P = \frac{1}{n}\mathbf{1}\mathbf{1}^T$; as $\mathcal{D}$ is regular $P\mathcal{A}(\mathcal{D}) = \mathcal{A}(\mathcal{D})P$ implies that $PL(\mathcal{D}) = L(\mathcal{D})P$. Now, $PB = B$ for all α; further $\mathbf{span}(I - \frac{1}{n}\mathbf{1}\mathbf{1}^T) = \mathbf{1}^\perp \subseteq \mathbf{span}(C^T)$ and so there exists a Z such that $P = I - ZC$. Hence as $L(\mathcal{D})$ is diagonalizable, statement 2. of Theorem 2 implies that the triple $(-L(\mathcal{D}), B, C)$ is output uncontrollable. ∎

B. Controlling Graphs

A special case of Theorem 2 can be formed when matrices A, B and C correspond to dynamics (3.3). In fact, this reduction produces a link between symmetry represented through the (signed) fractional automorphisms on $L(\mathcal{G})$ and the (output) controllability and stabilizability of the triple $(-L(\mathcal{G}), B(S), C(R))$.

Before proceeding to the corollary, we present some preliminaries. Building on the definition of *input symmetry* presented by Rahmani *et al.* [6], a matrix P is referred to as *symmetric* with respect to a family of vector $w_1, \ldots, w_m$ if $Pw_i = w_i$ for all $i = 1, \ldots, m$. Similarly, P is *asymmetric* if it is not symmetric. The result can now be summarized in the following corollary.

Corollary 5. *For R nonempty, consider one or more of the following conditions on P a nontrivial (signed) fractional automorphism on $L(\mathcal{G})$*

(a) P is input symmetric, i.e., $Pe_s = e_s$ for all $s \in S$.

(b) P is output asymmetric and non-output symmetric, i.e., $Pe_r \neq e_r$ for some $r \in R$ and $Pe_t = e_t$ for all $t \in V \setminus R$.

(c) P is stable symmetric, i.e., $\frac{1}{2}(P + P^T) \preceq I$ and $Pv_i = v_i$ for $\lambda_i(L(\mathcal{G})) > 0$.

Then there exists a $P \in \mathbb{R}^{n\times n}$ such that

1. (a) $\iff$ $(-L(\mathcal{G}), B(S))$ is uncontrollable.

2. (a) and (b) $\iff$ $(-L(\mathcal{G}), B(S), C(R))$ is output uncontrollable.

3. (a) and (c) $\iff$ $(-L(\mathcal{G}), B(S))$ is unstabilizable.

4. (a), (b) and (c) $\iff$ $(-L(\mathcal{G}), B(S), C(R))$ is output unstabilizable.

Further, if P is a fractional automorphisms then in (c), condition $\frac{1}{2}(P + P^T) \preceq I$ can be removed.

Proof: Firstly, as $L(\mathcal{G})$ is symmetric it is diagonalizable and for R nonempty, $C(R)$ is full row rank, satisfying the first conditions of Theorem 2. Consider $T = V \setminus R$, then $\begin{bmatrix} C(R)^T & C(T)^T \end{bmatrix}$ is unitary with $C(R)C(R)^T = I$ and $C(R)C(T)^T = 0$. Further, from the ($\Longleftarrow$) direction of the proof of Theorem 2, P can be constrained to $\mathbb{R}^{n\times n}$ as the eigenvectors of the undirected Laplacian are real. Similarly as $\mathbf{1}^T B(S) \neq 0$, i.e., the consensus mode associated with the eigenvalue 0 is always controllable, then P can be further constrained to those matrices satisfying $P\mathbf{1} = \mathbf{1}$ and $\mathbf{1}^T P = \mathbf{1}$, i.e., a matrix representing a signed fractional automorphism.

Condition (a) for P a nontrivial signed fractional automorphism is equivalent to $PB(S) = B(S)$ (condition (a) of Theorem 2).

Examining condition (b) of Theorem 2 and noting that $\begin{bmatrix} C(R)^T & C(T)^T \end{bmatrix}$ is invertible then

$$\begin{aligned} 0 &= P - I + ZC(R) \\ &= (P - I + ZC(R)) \begin{bmatrix} C(R)^T & C(T)^T \end{bmatrix} \\ &= \begin{bmatrix} (P-I)C(R)^T & (P-I)C(T)^T \end{bmatrix} \\ &\quad + \begin{bmatrix} ZC(R)C(R)^T & ZC(R)C(T)^T \end{bmatrix} \\ &= \begin{bmatrix} (P-I)C(R)^T & (P-I)C(T)^T \end{bmatrix} + \begin{bmatrix} Z & 0 \end{bmatrix}. \end{aligned}$$

Hence, condition (b) of Theorem 2 is then equivalent to $PC(R)^T = C(R)^T + Z$ and $PC(T)^T = C(T)^T$, i.e., output asymmetry and non-output symmetry.

As $P \in \mathbb{R}^{n\times n}$, A is symmetric and from condition (a), A and P^T commute, the second requirement of condition (c) simplifies to $\frac{1}{2}(P+P^T)L(\mathcal{G}) \succeq L(\mathcal{G})$. As $L(\mathcal{G})$ is diagonalizable this convex inequality is equivalent to $v_i^T \frac{1}{2}(P+P^T)L(\mathcal{G})v_i \geq v_i^T L(\mathcal{G})v_i$ for all eigenvectors v_i of $L(\mathcal{G})$. Or similarly, $\lambda_i(\mathcal{G})(v_i^T \frac{1}{2}(P+P^T)v_i) \geq \lambda_i(\mathcal{G})$. Now, as $\frac{1}{2}(P+P^T) \preceq I$ then $\alpha_i = v_i^T(\frac{1}{2}(P+P^T))v_i \leq 1$, with equality when $Pv_i = v_i$ and so $\lambda_i\alpha_i \geq \lambda_i$, which is true only if $\alpha_i = 1$ or $\lambda_i(\mathcal{G}) \leq 0$. Hence, $Pv_i = v_i$ when $\lambda_i(\mathcal{G}) > 0$ which corresponds to the stable modes of $-L(\mathcal{G})$.

Therefore, condition (a) and (c) is equivalent to condition (a) and (c) of Theorem 2, and so the result follows. Finally, if P is a fractional automorphism with $P_{ij} \in [0,1]$, it is a doubly stochastic matrix and as such $\frac{1}{2}(P+P^T)$ is also a doubly stochastic matrix with eigenvalues contained within the unit circle and as such always satisfies $\frac{1}{2}(P+P^T) \preceq I$. ■

Corollary 5 highlights the role of graph symmetries, described by signed fractional automorphisms, in the network controllability and stabilizability problem. We previously explored condition (a) in [13] presenting input symmetry with respect to signed fractional automorphisms as the generalization of Rahmani *et al.*'s controllability and leader symmetry link. Interestingly, condition (b) indicates that symmetry is not always detrimental to controllability. In fact for output controllability and stabilizability, symmetry is conducive over the output nodes R and asymmetry promotes output controllability and stabilizability when it occurs over the input and non-output nodes, i.e., over the set $S \bigcup (V \setminus R)$. For stabilizability, symmetry with respect to the stable modes of the system is important with stable symmetry implying unstabilizability from condition (c).

Similar to Theorem 2, Corollary 5 can be cast as a convex optimization problem. The output unstabilizability condition is equivalent to a non-identity solution P for the optimization problem:

$$\min_{P\in\mathbb{R}^{n\times n}, Z\in\mathbb{R}^{n\times p}} \mathbf{tr}(P) \tag{4.8}$$

$$\text{s.t. } L(\mathcal{G})P = PL(\mathcal{G}),\ PB = B,\ P = I - ZC$$

$$\frac{1}{2}(P+P^T) \preceq I,\ \frac{1}{2}(PL(\mathcal{G}) + L(\mathcal{G})P^T) \succeq L(\mathcal{G}) \tag{4.9}$$

$$\mathbf{1}^T P = \mathbf{1}^T,\ P\mathbf{1} = \mathbf{1}. \tag{4.10}$$

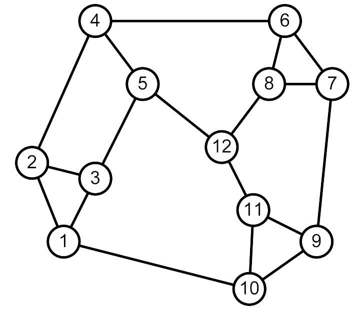


Figure 2: The asymmetric 3-regular Frucht graph.

An attraction of problem (4.8) over (4.5) is that (4.8) can be formulated on a smaller subspace by leveraging constraints (4.10) reducing the complexity of (4.5).

Interestingly, the matrix P in Corollary 5 can be further constrained to symmetric idempotent matrices i.e., matrices M such that $M^2 = M$ and $M = M^T$. This follows from the proof of Theorem 2 for symmetric A with $P = I - vv^T$ and $v^Tv = 1$. Some properties of an idempotent matrix $M \neq 0$ is that it is diagonalizable; its eigenvalues are either 0 or 1; the rank of the matrix is its trace; and $\|M\| \geq 1$ for any matrix norm $\|\cdot\|$ [17]. A symmetric assumption on P simplifies the constraints (4.9) in the optimization formulation to $P \preceq I$ and $(P-I)L(\mathcal{G}) \succeq 0$, respectively. The exploration of this idempotent property is left as future work.

An appeal of Corollary 5 is that graphs with very few standard symmetries - a small automorphism group - can still be analyzed structurally with respect to the *relaxed* symmetries that signed fractional automorphisms represent. Exploration over graphs that exhibit no traditional automorphisms with this machinery is particularly fruitful. To this end, Corollary 5 and the associated optimization problem (4.8) is applied to the asymmetric 3-regular Frucht graph.

Example 6. Let $\mathcal{G}$ be the Frucht graph displayed in Figure 2. Exercising the optimization problem (4.8) on every single input of $\mathcal{G}$ we find that single input nodes $T = \{1,2,6,10,11,12\}$ are controllable while the remaining single input nodes in the set $\overline{T} = \{3,4,5,7,8,9\}$ are uncontrollable. Examining the uncontrollable input node $R = \{3\}$, the pair $\left(-L(\mathcal{G}), R, \bar{R}\right)$ is output controllable for all single elements $\bar{R} \in V$. Consider reweighting a subset of the edges specifically $w_{23} = w_{47} = -1$, which introduces two unstable mode. The dynamics are stabilizable from all single inputs.

5. Conclusion

This paper presents a discussion on the links between symmetry and (output) controllability and stabilizability of networked dynamics. A convex optimization formulation of the problem has been presented that is applicable to both digraphs and graphs. We provided a necessary and sufficient condition on input, output, non-output and leader symmetry to render a network controllable and/or stabilizable. These conditions relied on a new notion of symmetry - a signed fractional automorphism. Future work of particular interest is examining other insights that the role of symmetry can bring to network control.

6. Appendix

Proof: **Theorem 2 ($\Longrightarrow$)**

(1) Let $Q = I - P$, assume condition (a) then $Q \neq 0$, $AQ = QA$ and $QB = 0$. Multiply $AQ = QA$ on the left by v_i^* for some $i \in 1, \dots, n$ then $v_1^* AQ = v_1^* QA$ and $\lambda_1(v_1^* Q) = (v_1^* Q)A$. As A is diagonalizable and so its eigenvectors are spanning for some v_i, say v_1, then $v_1^* Q \neq 0$ is a left eigenvector of A. Further, as $QB = 0$ then $v_1^* QB = 0$. Hence, from Proposition 1, condition (a) implies the pair (A, B) is uncontrollable.

(2) Assuming in addition condition (b), then $Q = ZC$ and letting $q^* = v_1^* Z$ then $q^* C$ is a left eigenvector of A with $q^* CB = 0$ and so from Proposition 1 the triple (A, B, C) is output uncontrollable.

(3) Assuming condition (a) and (c), as $2I - P - P^T \succeq 0$ then $Q + Q^T \succeq 0$. As A is diagonalizable its v_i's are spanning then for some i, say $i = 2$, we have $v_2^*(Q+Q^T)v_2 > 0$. Thus, $v_2^* Q \neq 0$ and following (1) is a left eigenvector of A with $v_2^* QB = 0$, i.e., v_2^* is an uncontrollable mode of A.

Further, as $A + A^T \succeq PA + (PA)^T$ then $QA + (QA)^T \succeq 0$, and multiplying on the left by v_2^* and the right by v_2 then

$$\begin{aligned} v_2^*(QA + (QA)^T)v_2 &\geq 0 \\ v_2^* QAv_2 + (v_2^* QAv_2)^* &\geq 0 \\ \lambda_2 v_2^* Qv_2 + \lambda_2^* v_2^* Q^T v_2 &\geq 0 \\ 2\text{Re}(\lambda_2 v_2^*(Q + Q^T)v_2) &\geq 0 \\ 2\text{Re}(\lambda_2)\text{Re}(v_2^*(Q + Q^T)v_2) &\geq 0, \end{aligned}$$

with the final inequality following from the fact that $v_2^*(Q + Q^T)v_2$ is real as $Q + Q^T$ is Hermitian. Hence, $\text{Re}(\lambda_2) \geq 0$. Therefore, v_2^* is an unstable mode and so by Proposition 1 the pair (A, B) is unstabilizable.

(4) Assuming conditions (b) in addition to (a) and (c) then letting $Q = ZC$ and $q^* = v_2^* Z$ similar the proof of (2) then by Proposition 1 the triple (A, B, C) is output unstabilizable.

Theorem 2 ($\Longleftarrow$)

(1) Assume (A, B) is uncontrollable and so there exists a v^* which is a left eigenvector of A such that $v^* B = 0$ and the associated right eigenvalue/eigenvector is (λ, w) with $v^* w = 1$ (from Proposition 1). Consequently, v^T is also an uncontrollable mode as $0 = \overline{v^* B} = v^T B$ with associated right eigenvalue/eigenvector $\left(\bar{\lambda}, \bar{w}\right)$ with $v^T \bar{w} = 1$. Let $P = I - \frac{1}{2}\left(wv^* + \bar{w}v^T\right) = I - \text{Re}(wv^*)$ which is a non-identity real matrix, then

$$\begin{aligned} &AP - PA \\ &= A(I - \frac{1}{2}(wv^* + \bar{w}v^T)) - (I - \frac{1}{2}(wv^* + \bar{w}v^T))A \\ &= A - \frac{1}{2}\lambda wv^* - \frac{1}{2}\bar{\lambda}\bar{w}v^T - A + \frac{1}{2}\lambda wv^* + \frac{1}{2}\bar{\lambda}\bar{w}v^T = 0, \end{aligned}$$

and $PB = (I - \frac{1}{2}wv^* - \frac{1}{2}\bar{w}v^T)B = B$. Therefore, P satisfies condition (a).

(2) Assume (A, B, C) is output uncontrollable then following the proof of (1), v^* can be selected such that $v^* = q^* C$ and so $v^T = q^T C$. Hence, $P = I - \frac{1}{2}(wv^* + \bar{w}v^T) = I - \frac{1}{2}wq^* C - \frac{1}{2}\bar{w}q^T C$, and $Z = \frac{1}{2}(wq^* + \bar{w}q^T) = \text{Re}(wq^*) \in \mathbb{R}^{n \times n}$. Hence, P satisfies condition (a) and (b).

(3) Assume A is symmetric and (A, B) is unstabilizable then following the proof of (1), v^* can be selected such that $\text{Re}(\lambda) \geq 0$. Now, as A is real symmetric then $v^* = w^T$ and λ is real and $P = I - ww^T$. Therefore, $2I - P - P^T = ww^T + (ww^T)^T$ and as ww^T is positive semi-definite then $2I - P - P^T \succeq 0$. Finally,

$$\begin{aligned} &A + A^T - PA - (PA)^T \\ &= A + A^T - (I - ww^T)A - ((I - ww^T)A)^T \\ &= ww^T A + (ww^T A)^T = \lambda ww^T + \lambda(ww^T)^T \\ &= 2\lambda ww^T \succeq 0, \end{aligned}$$

Hence, P satisfies condition (a) and (c).

(4) Follows the proof of (2). ■

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