Controllability and Observability of Network-of-Networks via Cartesian Products

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***Abstract*—The paper presents a system theoretic analysis framework for a network-of-networks, formed from smaller factor networks via graph Cartesian products. We provide a compositional framework for extending the controllability and observability of the factor networks to that of the composite network-of-networks. We then delve into the effectiveness of designing control and estimation algorithms for the composite network via its symmetry and gramian structure. An example demonstrating the usefulness of our results in the context of social networks with a Cartesian product structure is then presented.**

***Index Terms*—Composite networks, coordination algorithms, graph Cartesian product, network controllability, network observability.**

I. Introduction

ONE of the central themes in control theory pertains to the analysis of properties that are preserved under composition of systems. A prime example of this is the use of passivity as a system property, preserved under feedback and parallel interconnections, with a multitude of ramifications for system synthesis [2]. Other examples include forming stable interconnected systems using stable atomic subsystems via the small-gain theorem, composite Lyapunov functions, and in the framework of compartmental systems [3], as well as establishing controllability and observability of series-parallel and feedback systems in terms of controllability and observability of their subsystems [4]. Decomposition has been yet another facet of system analysis via a compositional perspective. Decomposition techniques provide compact descriptions of systems, providing both analysis and computational benefits. Jordan decomposition, for example, reveals the finite zero structure as well as the invariance properties of linear systems [5]. Other examples of the decomposition approach include the Kalman decomposition and structural decomposition methods [6], [7]. This latter class of representations have implications for sensor and actuator selection and placement problems [8].

The focus of the present paper is on a network-centric compositional theory for networked dynamic systems [1]. Networked dynamic systems are an integral part of the technological world including the power grids and information networks, as well as in areas such as biological and social systems. An explosion of research in the area of networked systems has eventuated [9]–[11]. In order to provide a concrete utility of a compositional theory of networked systems, in this paper we focus on arguably the most basic property of controlled processes, namely, their controllability and observability.[1]

Controllability and observability of networked dynamic systems adopting consensus-type coordination algorithms have recently attracted the attention of researchers in a multitude disciplines [12]–[15]. Network controllability becomes important when a networked system is influenced or observed by an external entity; such scenarios include networked robotics [16], [17], human-swarm interactions [18], network security [19], [20], and quantum networks [21]. More specifically, in this paper we develop a compositional theory for controllability and observability of certain classes of linear systems operating over networks that are obtained from the Cartesian product of smaller factor networks; we also refer to these interconnections as *network-of-networks*. We then proceed to present two schemes, dubbed *p-simple* and *layered control*, for extending controllability (and by duality, observability) of the factors to the controllability of the composite network.

The schemes are relevant to dynamics driven by a large class of network-based matrices, including the Laplacian and adjacency matrices. The minimum number of required control inputs (respectively, outputs) for controllability (respectively, observability) is then addressed by exploiting the automorphism group of the composite network. Furthermore, the effectiveness of the layered control is investigated by relating properties of the controllability Gramian of the factors to that of the composite network.

Our results are applicable to control problems with an underlying layered structure forming a Cartesian product. These types of systems are induced by, or eventuate through, many processes. Man-made structures, for instance, are fabricated with such a layered form, with applications including fault detection

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[1]In the meantime, it will be evident from the subsequent discussion that other system properties can be approached from such a compositional perspective.

in infrastructure networks [22], piezoelectric sensor and actuator placement in smart structures [23], [24] and efficient control of quantum computing networks [21], [25]. Layered structures are often a by-product of the problem formulation itself such as the uniform discretization of PDEs, and as such our theory has application to the control and monitoring of fluid and heat flow [26], [27]. Cartesian products also appear as a result of the network analysis process, for example through the classification of entities in a network. This occurs in the analysis and influence of opinion dynamics on social networks with nodes representing people with common interest groups [28]–[31].

The organization of the paper is as follows. We begin by introducing relevant background material pertaining to graphs, Cartesian products, and Kronecker products. This is followed by introducing a linear network-based state dynamics that is applicable to a large class of systems driven by matrix representations of graphs. We then provide a control scheme that forms a controllable composite graph from its controllable factors. Symmetries in the graph provide a sufficient condition for the control scheme to produce the minimal number of inputs for a controllable network. A second control scheme extends the controllability of the factor graphs to that of the composite graph by exploiting the layering structure of the Cartesian product. The effectiveness of the control for layered networks is subsequently quantified using the controllability Gramian. A layered output feedback controller is then presented pertaining to the layered control scheme. Finally, the opinion state of a Cartesian social network is estimated by observing a subset of nodes, demonstrating the utility of the observability properties of a Cartesian network in terms of its factors.

II. Notation and Background

In this section, we provide the notation, and a brief background on constructs and models that will be used throughout the paper.

The zero vector and the vector of ones are denoted, respectively, as $\mathbf{0} := [0, 0, \ldots, 0]^T$ and $\mathbf{1} := [1, 1, \ldots, 1]^T$. For the column vector $v \in \mathbb{R}^p$, both v_i and $[v]_i$ denote its ith element. For the matrix $M \in \mathbb{R}^{p\times q}$, $[M]_{ij}$ denotes the entry in its ith row and jth column. The $n \times n$ identity matrix is denoted by I_n, and e_i is the column vector with all zero entries except for $[e_i]_i = 1$. For matrices $M, N \in \mathbb{R}^{n\times n}$, we write $M \succeq 0$ when M is a positive semidefinite matrix and $M \succeq N$ if $M - N \succeq 0$. The real part of the eigenvalue λ of the matrix M is denoted by $\mathbf{Re}\{\lambda(M)\}$ and the maximum of this value over the spectrum of M by $\max \mathbf{Re}\{\lambda(M)\}$. The cardinality of a set Z is denoted by $|Z|$. For matrices $M \in \mathbb{R}^{n\times p}$ and $V \in \mathbb{R}^{n\times q}$, the notation $M \perp V$ indicates that $v^T M = 0$ for some nonzero v in the column space of V, similarly, $M \not\perp V$ indicates that no such v exists.

A. Graphs

A weighted digraph $\mathcal{G} = (V, E, W)$ is characterized by a node set V with cardinality n, an edge set E comprised of ordered pairs of nodes with cardinality m, and a weight set W with cardinality m, where an edge exists from node i to j if $(i, j) \in E$ with edge weight $w_{ji} \in W$. The adjacency matrix

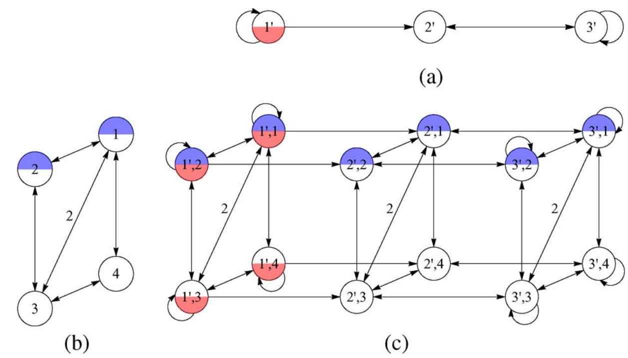

Fig. 1. Factor graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ and composite graph $\mathcal{G}_1 \square \mathcal{G}_2$. Edge weights of all graphs are 1 unless otherwise marked. The shading on the nodes pertains to Example IV-5, IV-15 and IV-18. (a) $\mathcal{G}_1$; (b) $\mathcal{G}_2$; (c) $\mathcal{G} = \mathcal{G}_1 \square \mathcal{G}_2$.

of the digraph $\mathcal{G}$ denoted as $\mathscr{A}(\mathcal{G})$, is an $n \times n$ matrix with $[\mathscr{A}(\mathcal{G})]_{ij} = w_{ij}$ when $(j, i) \in E$ and $[\mathscr{A}(\mathcal{G})]_{ij} = 0$ otherwise. The self-loop matrix $\mathscr{D}_s(\mathcal{G}) \in \mathbb{R}^{n\times n}$ is a diagonal matrix with w_{ii} as its ith diagonal entry. The in-degree matrix $\mathscr{D}_{in}(\mathcal{G}) \in \mathbb{R}^{n\times n}$ is a diagonal matrix with the number of edges incident to node i (its in-degree) as its ith diagonal entry; the out-degree matrix $\mathscr{D}_{out}(\mathcal{G})$ is similarly defined. Other graph-theoretic terminology used in the paper can be found in [17].

Special types of digraphs include undirected graphs where $(i, j) \in E$ implies $(j, i) \in E$, unweighted graphs where $w_{ji} = 1$ for all $(i, j) \in E$. Examples include the undirected n-node path graph, denoted as $\mathcal{P}_n = (V, E, W)$, with edge set $(i, j) \in E$ and $w_{ji} = 1$ if and only if $|i - j| = 1$, and the undirected n-node cycle graph, denoted $\mathcal{C}_n$, formed by adding edges $(1, n)$ and $(n, 1)$ with weights $w_{1n} = w_{n1} = 1$ to the n-node path graph.

B. Cartesian Products

There are a number of ways to synthesize large-scale networks from a set of smaller size graphs [32]. The Cartesian product is one such method and is defined for a pair of *factor* graphs $\mathcal{G}_1 = (V_1, E_1, W_1)$ and $\mathcal{G}_2 = (V_2, E_2, W_2)$ and denoted by $\mathcal{G} = \mathcal{G}_1 \square \mathcal{G}_2$. The (Cartesian) product graph $\mathcal{G}$ has the vertex set $V_1 \times V_2$, and there is an edge from vertex (i, p) to (j, q) in $V_1 \times V_2$ if and only if either $i = j$ and (p, q) is an edge in E_2, or $p = q$ and (i, j) is an edge in E_1. The corresponding weight if an edge exists is $w_{((j,q),(i,p))} = \delta_{pq} w_{ji} + \delta_{ij} w_{qp}$, where $\delta_{uv} = 1$ if $u = v$ and 0 if otherwise. An example of a Cartesian product of two *factor* graphs is displayed in Fig. 1. We note that the Cartesian product operation is commutative and associative, i.e., the products $\mathcal{G}_1 \square \mathcal{G}_2$ and $\mathcal{G}_2 \square \mathcal{G}_1$ are isomorphic; similarly $(\mathcal{G}_1 \square \mathcal{G}_2) \square \mathcal{G}_3$ and $\mathcal{G}_1 \square (\mathcal{G}_2 \square \mathcal{G}_3)$ are isomorphic.

A graph is called *prime* if it cannot be decomposed into the product of non-trivial graphs; otherwise a graph is referred to as composite. All graphs have a *prime factor decomposition* of the form $\mathcal{G}_1^{k_1} \square \cdots \square \mathcal{G}_m^{k_m}$, where $\mathcal{G}_i$ is prime for all i and $\mathcal{G}_i^{k_i}$ denotes k_i Cartesian products of $\mathcal{G}_i$. Sabidussi [33] and Vizing [34] highlighted the fundamental properties of prime graphs, noting that all connected graphs decompose uniquely into primes, up to a reordering. Further, Feigenbaum [35] demonstrated that a digraph can be factored into primes in polynomial-time.

C. Kronecker Products

Kronecker algebra is one of the main constructs used in this paper. The key results are presented here and we refer the reader to [36] for a more detailed treatment.

Let $M = [m_{ij}] \in \mathbb{R}^{r\times s}$ and $N \in \mathbb{R}^{p\times q}$. The Kronecker product $M \otimes N$ can be formed by replacing the ijth entry of M by the matrix $m_{ij}N$, for every i and j. Hence

$$M \otimes N = \begin{bmatrix} m_{11}N & \cdots & m_{1s}N \\ \vdots & & \vdots \\ m_{r1}N & \cdots & m_{rs}N \end{bmatrix} \in \mathbb{R}^{rp\times sq}.$$

The Kronecker products $M \otimes N$ and $N \otimes M$ are permutation equivalent, i.e., there exist permutation matrices Π and Υ such that $M \otimes N = \Pi(N \otimes M)\Upsilon$, where if M and N are square, $\Pi = \Upsilon^T$. Further, the Kronecker product exhibits the mixed-product property, $(M \otimes N)(R \otimes S) = MR \otimes NS$, where $M \in \mathbb{R}^{r\times s}$, $N \in \mathbb{R}^{p\times q}$, $R \in \mathbb{R}^{s\times k}$, and $S \in \mathbb{R}^{q\times l}$.

Of particular interest to our presentation is the Kronecker sum defined on square matrices $M \in \mathbb{R}^{s\times s}$ and $N \in \mathbb{R}^{r\times r}$ as $M \oplus N := M \otimes I_r + I_s \otimes N$. A property of the Kronecker sum is that given (either left or right) eigenvalue-eigenvector pairs of M and N as (λ_i, u_i) for $i = 1, \ldots, n$ and (μ_j, v_j) for $j = 1, \ldots, m$, respectively, then $(\lambda_i + \mu_j, u_i \otimes v_j)$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ are eigenvalue-eigenvector pairs of $M \oplus N$ [36]. Let us denote the eigenvalue $\lambda_i + \mu_j$ of $M \oplus N$ by ϕ_{ij}, where λ_i and μ_j are, respectively, the eigenvalues of M and N, with the corresponding eigenvectors u_i and v_i. If M and N are diagonalizable, then the set of eigenvectors $u_i \otimes v_j$ are spanning over $\mathbb{R}^{rs}$. Hence, (ϕ, w) is an eigenvalue-eigenvector pair for $M \oplus N$ if and only if $w = \sum_{(i,j)\in\Phi} u_i \otimes v_j$ for some distinct eigenvectors u_i and v_j of M and N, and $\phi = \phi_{ij}$ for all $(i, j) \in \Phi$; in this case ϕ has multiplicity $|\Phi|$.

Lastly, we mention that the Kronecker sum satisfies the matrix exponential property

$$e^{M\oplus N} = e^M \otimes e^N. \tag{1}$$

D. Graph Automorphisms

A graph automorphism is a permutation σ on the vertex set of the graph such that $\mathcal{G}$ contains an edge (i, j) with weight w_{ji} if and only if it also contains an edge $(\sigma(i), \sigma(j))$ with weight $w_{\sigma(j)\sigma(i)} = w_{ji}$. The set of automorphisms of $\mathcal{G}$ (which forms a group under composition) is denoted $\mathrm{Aut}(\mathcal{G})$. Every automorphism can be represented uniquely as a permutation matrix J that commutes with the adjacency matrix of $\mathcal{G}$, i.e., $J\mathscr{A}(\mathcal{G}) = \mathscr{A}(\mathcal{G})J$.

III. Problem Setup

There are a number of ways to construct the (system) matrix $A(\mathcal{G}) \in \mathbb{R}^{n\times n}$ associated with an n node graph $\mathcal{G}$. One option is the adjacency matrix $\mathscr{A}(\mathcal{G})$ which we have already touched upon in Section II-A. The adjacency matrix uniquely defines the graph $\mathcal{G}$. In the meantime, there are other matrix representations of the graph that do not necessary uniquely define the graph—for example—the self-loop matrix $\mathscr{D}_s(\mathcal{G})$, the in-degree matrix $\mathscr{D}_{in}(\mathcal{G})$, and out-degree matrix $\mathscr{D}_{out}(\mathcal{G})$. Further, one of the properties common to all the aforementioned matrix representations is that they are *symmetry preserving*: the representation $A(\cdot)$ is such that for all $\sigma \in \mathrm{Aut}(\mathcal{G})$, with the corresponding permutation matrix J, $A(\mathcal{G})J = JA(\mathcal{G})$.

In this paper, we will consider networks with the system matrix belonging to the family of symmetry preserving representations of the form[2]

$$[A(\mathcal{G})]_{ij} = \begin{cases} rw_{ii} + \sum_{j\neq i} f(w_{ij}, w_{ji}) & \text{for } i = j \\ g(w_{ij}, w_{ji}) & \text{otherwise} \end{cases} \tag{2}$$

where $r \in \mathbb{R}$, and $f(\cdot)$ and $g(\cdot)$ are real-valued functions such that $f(0, x) = g(0, x) = 0$. We denote this family of matrix representations by $\mathbf{A}_\oplus$.

All matrix representations of digraphs previously mentioned are members of $\mathbf{A}_\oplus$. Other members of $\mathbf{A}_\oplus$ include the in-degree graph Laplacian (or Laplacian) $\mathscr{L}(\mathcal{G})$, defined as $[\mathscr{L}(\mathcal{G})]_{ij} = -[\mathscr{A}(\mathcal{G})]_{ij}$ for $i \neq j$ and $[\mathscr{L}(\mathcal{G})]_{ii} = [\mathscr{D}_{in}(\mathcal{G})]_{ii}$ for all i. The Laplacian matrix is featured in the popular consensus dynamics; with the exception of self-loops, it uniquely defines the graph $\mathcal{G}$. Other noteworthy members of $\mathbf{A}_\oplus$ are the out-degree graph Laplacian $\mathscr{L}_{out}(\mathcal{G})$ defined as $[\mathscr{L}_{out}(\mathcal{G})]_{ij} = -[\mathscr{A}(\mathcal{G})]_{ij}$ for $i \neq j$ and $[\mathscr{L}_{out}(\mathcal{G})]_{ii} = [\mathscr{D}_{out}(\mathcal{G})]_{ii}$ for all i, used in the advection dynamics [37], and the M-matrix representation of graphs investigated in [38]. The set $\mathbf{A}_\oplus$ also contains other matrix representations of graphs, some of which will be featured in the paper. For example, it is straightforward to show that $\mathbf{A}_\oplus$ is closed under addition, providing a simple mechanism to generate new members, e.g., matrices that encode diffusion-advection processes on graphs.

A special feature of $\mathbf{A}_\oplus$, formally shown in Proposition A.1 of the Appendix, is that as well as being symmetry preserving these representations also satisfy the identity

$$A(\mathcal{G}_1 \square\, \mathcal{G}_2) = A(\mathcal{G}_1) \oplus A(\mathcal{G}_2)$$

for all graphs $\mathcal{G}_1$ and $\mathcal{G}_2$. This identity serves as a key observation for examining dynamics on networks that admit non-trivial Cartesian factorizations.

In this paper, we will be exploring a compositional theory for the controllability and observability of factor systems

$$\dot{x}_i(t) = A(\mathcal{G}_i)x_i(t) + B_i u_i(t), \quad y_i(t) = C_i x_i(t)$$

for $i = 1, \ldots, s$, where $A(\cdot) \in \mathbf{A}_\oplus$. Composed together to form the composite system

$$\dot{x}(t) = A\left(\prod_{\square} \mathcal{G}_i\right) x(t) + \prod_{\otimes} B_i u(t), \quad y(t) = \prod_{\otimes} C_i x_i(t). \tag{3}$$

We often refer to such dynamics by specifying the triplet $(A(\mathcal{G}), B, C)$, or if only the inputs and outputs are of interest, by matrix pairs $(A(\mathcal{G}), B)$ and $(C, A(\mathcal{G}))$, respectively. The input and output matrices B and C can also encode an incidence relationship on the graph. If the set of input nodes in an n

[2]Here we assume that if there is no edge $(i, j) \in E$ then $w_{ji} = 0$.

node graph is $S = \{i_1, i_2, \dots, i_p\}$ for $i_1 < i_2 < \cdots < i_p$, the corresponding input matrix is $B = [e_{i_1}, e_{i_2}, \dots, e_{i_p}] \in \mathbb{R}^{n\times p}$. We denote the input and output matrices generated in this manner as $B_n(S)$ and $C_n(S) := B_n(S)^T$; when n is clear from the context, it is omitted for brevity. Consider now the case where $B_1 := B(S_1)$ and $B_2 := B(S_2)$; then for $S = S_1 \times S_2 := \{(i,j) | i \in S_1, j \in S_2\}$ we have $B(S) = B_1 \otimes B_2$. Hence, extending to the composition of s input sets, as in (3), then $\prod_\otimes B(S_i) = B(\prod_\times S_i)$.

Remark III.1: For the discrete-time version of (3)

$$x(k+1) = A(\mathcal{G})x(k) + Bu(k), \quad y(k) = Cx(k)$$

the analogous symmetry preserving representations satisfy

$$A(\mathcal{G}_1 \Box \mathcal{G}_2) = A(\mathcal{G}_1) \otimes A(\mathcal{G}_2).$$

We denote this family of matrix representations as $\mathbf{A}_\otimes$. The matrix exponential of the matrix representations in $\mathbf{A}_\oplus$ are members of $\mathbf{A}_\otimes$ (this follows from (1)). Note that the set $\mathbf{A}_\otimes$ is closed under multiplication.

IV. Controllability of Cartesian Product Networks

In this section, we delve into the problem of characterizing conditions under which controllability of a composite network can be implied from the controllability of its factors, with respect to the Cartesian product. Due to linear systems duality, the results of this paper will be presented in terms of network controllability, but are equally applicable to network observability problems. Our analysis is presented in term of two factors dynamics defined by the pairs $(A(\mathcal{G}_1), B_1)$ and $(A(\mathcal{G}_2), B_2)$, and composed to form the composite network dynamics defined by $(A(\mathcal{G}_1 \Box \mathcal{G}_2), B_1 \otimes B_2)$, and can be extended to larger numbers of factors, by sequential composition.

One of the cornerstones of our analysis in this venue is the properties of Kronecker products in conjunction with the Popov–Belevitch–Hautus (PBH) test which states that the pair (A, B) is uncontrollable if and only if there exists a left eigenvalue-eigenvector pair (λ, v) of A such that $v^T B = 0$ [39]. The PBH test implies that the pair $(A(\mathcal{G}_1 \Box \mathcal{G}_2), B_1 \otimes B_2)$ is uncontrollable if and only if $A(\mathcal{G}_1 \Box \mathcal{G}_2)$ admits a left eigenvector that is orthogonal to $B_1 \otimes B_2$. In the meantime as discussed in Section II-C, the left eigenvectors of a diagonalizable $A(\mathcal{G}_1 \Box \mathcal{G}_2)$ are of the form

$$\sum_k u_k \otimes v_k$$

where u_k's and v_k's are the left eigenvectors of $A(\mathcal{G}_1)$ and $A(\mathcal{G}_2)$, respectively. Thereby, the pair $(A(\mathcal{G}_1 \Box \mathcal{G}_2), B_1 \otimes B_2)$ is uncontrollable if and only if, for some left eigenvectors u_k and v_k for $A(\mathcal{G}_1)$ and v_k of $A(\mathcal{G}_2)$, respectively, we have

$$\left(\sum_k u_k \otimes v_k\right)^T (B_1 \otimes B_2) = \left(\sum_k u_k^T \otimes v_k^T\right)(B_1 \otimes B_2) = \sum_k \left(u_k^T B_1\right) \otimes \left(v_k^T B_2\right) = 0.$$

In this venue, the following result is particularly illuminating and whose converse is the core to our first theorem.

Theorem 1 ([40]): If $u_1, u_2, \dots, u_n$ are linearly independent vectors and $v_1, v_2, \dots, v_n$ are arbitrary vectors, then

$$\sum_{k=1}^n u_k \otimes v_k = 0 \quad \text{implies that} \quad v_k = 0 \quad \text{for all } k.$$

Moreover, the roles of u_k's and v_k's in the above statement can be reversed.

A. Controllability Induced by the Cartesian Product

The following theorem introduces our first method via which the controllability of the composite graph can directly be deciphered from the controllability of its factors.

Theorem IV.1: Let $A(\cdot) \in \boldsymbol{A}_\oplus$, $\mathcal{G} = \mathcal{G}_1 \Box \mathcal{G}_2$, and $B = B_1 \otimes B_2$. Then if $A(\mathcal{G})$ is diagonalizable, the pair $(A(\mathcal{G}), B)$ is controllable if and only if

1) the pairs $(A(\mathcal{G}_1), B_1)$ and $(A(\mathcal{G}_2), B_2)$ are controllable and
2) $B_1 \not\perp [U_1, U_2, \dots, U_p]$ or $B_2 \not\perp [V_1, V_2, \dots, V_p]$, where the columns of U_i are the orthogonal left eigenvectors of eigenvalues $\tilde{\lambda}_i$ of $A(\mathcal{G}_1)$ (similarly for pairs $(\tilde{\mu}_i, V_i)$ of $A(\mathcal{G}_2)$), such that $\tilde{\lambda}_1 + \tilde{\mu}_1 = \tilde{\lambda}_2 + \tilde{\mu}_2 = \ldots = \tilde{\lambda}_p + \tilde{\mu}_p$, $\tilde{\lambda}_i \neq \tilde{\lambda}_j$ for all $i \neq j$, and $p > 1$.

Proof: As $A(\mathcal{G}_1 \Box \mathcal{G}_2)$ is diagonalizable all its eigenvectors take the form $\sum u_i \otimes v_j$, where u_i and v_j are eigenvectors of $A(\mathcal{G}_1)$ and $A(\mathcal{G}_2)$, respectively. Consider an eigenvalue $A(\mathcal{G}_1 \Box \mathcal{G}_2)$, expressed as $\tilde{\lambda}_1 + \tilde{\mu}_1$ in terms of the set of distinct eigenvalues $\tilde{\lambda}_i$'s and $\tilde{\mu}_i$'s of $A(\mathcal{G}_1)$ and $A(\mathcal{G}_2)$, respectively. Then if

$$\tilde{\lambda}_1 + \tilde{\mu}_1 = \tilde{\lambda}_2 + \tilde{\mu}_2 = \ldots = \tilde{\lambda}_p + \tilde{\mu}_p$$

the left eigenvectors of $\tilde{\lambda}_1 + \tilde{\mu}_1$ form the basis set $[U_1 \otimes V_1 \; U_2 \otimes V_2 \; \cdots \; U_p \otimes V_p]$. Now $(A(\mathcal{G}_1 \Box \mathcal{G}_2), B_1 \otimes B_2)$ is uncontrollable if and only if there exists an eigenvalue $\tilde{\lambda}_1 + \tilde{\mu}_1$ with $B_1 \otimes B_2 \perp [U_1 \otimes V_1 \; U_2 \otimes V_2 \; \cdots \; U_p \otimes V_p]$, or equivalently, there exists an eigenvector $\sum_{i=1}^p u_i \otimes v_i$ for some u_i in the span of U_i (similarly for v_i in V_i) such that $\sum_{i=1}^p u_i^T B_1 \otimes v_i^T B_2 = 0$. This occurs if and only if either

1) $B_1 \perp U_i$ or $B_2 \perp V_i$ for some i, i.e, $(A(\mathcal{G}_1), B_1)$ or $(A(\mathcal{G}_2), B_2)$ is uncontrollable, or
2) $B_1 \not\perp U_i$ and $B_2 \not\perp V_i$ for all i and $p > 1$. In other terms, $u_i^T B_1 \neq 0$ and $v_j^T B_2 \neq 0$ for all i. In this case, from the converse of Theorem 1, (2) implies that $u_1^T B_1, u_2^T B_1, \dots, u_p^T B_1$ are linearly dependent vectors and $v_1^T B_2, v_2^T B_2, \dots, v_p^T B_2$ are linearly dependent vectors, or equivalently $B_1 \perp [U_1, U_2, \dots, U_p]$ and $B_2 \perp [V_1, V_2, \dots, V_p]$.

Conversely, if $B_1 \perp [U_1, U_2, \dots, U_p]$, for some nonzero u in the span of $[U_1, U_2, \dots, U_p]$, $u^T B_1 = 0$, $(u \otimes v)^T (B_1 \otimes B_2) = 0$ for all v. Similarly if $B_2 \perp [V_1, V_2, \dots, V_p]$, there exists a nonzero v such that $(u \otimes v)^T (B_1 \otimes B_2) = 0$ for all u. ■

The following example demonstrates the utility of this result.

Example IV.2: Define the matrix representation $A(\mathcal{G}) = \mathscr{A}(\mathcal{G}) - \mathscr{D}_s(\mathcal{G})$. As $\boldsymbol{A}_\oplus$ is closed under addition $A(\cdot) \in \boldsymbol{A}_\oplus$. Consider graphs $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G} = \mathcal{G}_1 \square \mathcal{G}_2$ in Fig. 1. The eigenvalues of $A(\mathcal{G}_1)$ and $A(\mathcal{G}_2)$ are $\{\lambda_i\} = \{-1, 0, 1\}$ and $\{\mu_i\} = \{-2, 1-\sqrt{5}, 0, 1+\sqrt{5})$, respectively, with associated left eigenvectors $\{u_i\}$ and $\{v_i\}$, respectively. For $S_1 = \{1'\}$ and $S_2 = \{1, 2\}$, the pairs $(A(\mathcal{G}_1), B_3(S_1))$ and $(A(\mathcal{G}_2), B_4(S_2))$ are controllable. The nodes corresponding to sets S_1 and S_2 are half shaded in Fig. 1. Consider $B_{12}(S) = B_3(S_1) \otimes B_4(S_2)$ where $S = \{(1', 1), (1', 2)\}$, denoted by full shaded nodes in Fig. 1. As $\lambda_i + \mu_j$ for all i, j are the eigenvalues of $A(\mathcal{G})$, it is efficient to check that the eigenvalues are all distinct except $\lambda_3 + \mu_1 = \lambda_1 + \mu_3$ with $[v_1, v_3] = [e_1 - e_3, e_2 - e_4]$. As $B_4(S_2) \not\perp [v_1, v_3]$, from Theorem IV.1, it follows that $(A(\mathcal{G}), B_{12}(S))$ is controllable.

It is clear from property (1) of Theorem IV.1 that factor system controllability is necessary for composite controllability. The natural question is whether there are conditions under which property (1) is also sufficient for controllability so property (2) does not need to be formally checked. In such a scenario, uncontrollability of $(A(\mathcal{G}_1 \square \mathcal{G}_2), B_1 \otimes B_2)$ can only be induced by an uncontrollable factor.

There are two broad classes of network realizations for which the answer to this question is affirmative; these cases are singled out and discussed below under the headings of *p-simple* and *layered control*, by restricting the eigenvalue multiplicities of $A(\mathcal{G}_1 \square \mathcal{G}_2)$ and the form of the input matrices, respectively.

Before we delve into discussing these two special realizations, however, we expand upon why in general, an uncontrollable Cartesian product network might arise from two controllable factors, and highlight the requirement on diagonalizability.

B. Uncontrollability and Non-Diagonalizability

If the composition of two controllable networks is uncontrollable, it is the case that the operation of taking the Cartesian product has led to an uncontrollable mode, absent in either factors. From property (2) of Theorem IV.1, this mode (or eigenvector) is formed from the introduction of eigenvalue multiplicities due to taking the sum of distinct eigenvalues. This case is illustrated in the following example.

Example IV.3: The matrix $\mathscr{L}(\mathcal{P}_2)$ has simple eigenvalues and for $S = \{1\}$ the pair $(-\mathscr{L}(\mathcal{P}_2), B_2(S))$ is controllable. Furthermore, $\mathcal{P}_2 \square \mathcal{P}_2 = \mathcal{C}_4$, a length four cycle graph, with one repeated eigenvalue, and $B_2(S) \otimes B_2(S) = B_4(S)$. But all cycle graphs are uncontrollable from one node (see [14], [41]); thus $(-\mathscr{L}(\mathcal{C}_4), B_4(S))$ is uncontrollable.

The eigenvalue multiplicity property of $A(\mathcal{G}_1 \square \mathcal{G}_2)$ as required by property (2) of Theorem IV.1 ensures that an uncontrollable mode for the composite network can be accounted for by the corresponding uncontrollable mode in one of its factors.

The theorem's diagonalizability requirement stems from a similar source. For non-diagonalizable $A(\mathcal{G}_1 \square \mathcal{G}_2)$, uncontrollable modes can arise in the composite eigenspace not formed from linear combinations of the Kronecker product of the factors' eigenspaces. Thus for a non-diagonalizable $A(\mathcal{G}_1 \square \mathcal{G}_2)$, Theorem IV.1 is necessary but not sufficient for composite controllability, an illustration of this follows.

Example IV.4: Denote the two node graph containing a single directed edge (2,1) as $\mathscr{D}_2$. Then $\mathscr{A}(\mathscr{D}_2) = [0, e_1]$ is non-diagonalizable, with two zero eigenvalues, and for $S = \{2\}$, the pair $(\mathscr{A}(\mathscr{D}_2), B_2(S))$ is controllable. Further, let $\mathscr{D} = \mathscr{D}_2 \square \mathscr{D}_2$ and $B_4(S) = B_2(S) \otimes B_2(S)$. Now, $\mathscr{A}(\mathscr{D}) = [0, e_1, e_1, e_2 + e_3]$ has four zero eigenvalues, and so the pair $(\mathscr{A}(\mathscr{D}), B_4(S))$ satisfies properties (1) and (2) of Theorem IV.1. But, $\mathscr{A}(\mathscr{D})$ has a left eigenvector $v = e_2 - e_3$ and so $v^T B_4(S) = 0$; thus $(\mathscr{A}(\mathscr{D}), B_4(S))$ is uncontrollable.

C. p-Simple and Symmetry

From Theorem IV.1, if the sum of all distinct eigenvalues of the factors is unique then property (1) is necessary *and sufficient* for the controllability of the composite network. We refer to $A(\mathcal{G}_1 \square \mathcal{G}_2)$ with this feature as *p-simple* with respect to $A(\mathcal{G}_1)$ and $A(\mathcal{G}_2)$, or when these factors are clear from the context just p-simple. The motivation of this terminology is that if $A(\mathcal{G})$ has only simple eigenvalues then for any factorization, factor's eigenvalues are unique and combine uniquely, i.e., $A(\mathcal{G})$ is p-simple with respect to *any* factorization. An example of Theorem IV.1 applied to a p-simple composite graph follows.

Example IV.5: Consider the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ in Fig. 1. Let the eigenvalues of $\mathscr{L}(\mathcal{G}_1)$ be $\lambda_1, \lambda_2, \lambda_3$ and similarly the eigenvalues for $\mathscr{L}(\mathcal{G}_2)$ be $\mu_1, \mu_2, \mu_3, \mu_4$. Further as $\lambda_i + \mu_j$ for $i = 1, 2, 3$ and $j = 1, \ldots, 4$ are the eigenvalues of $\mathscr{L}(\mathcal{G})$, by checking that $\lambda_i + \mu_j \neq \lambda_s + \mu_t$ for all $(i, j) \neq (s, t)$, the eigenvalues can be verified to be distinct, and so $\mathscr{L}(\mathcal{G})$ is p-simple. For $S_1 = \{1'\}$ and $S_2 = \{1, 2\}$, the pairs $(-\mathscr{L}(\mathcal{G}_1), B_3(S_1))$ and $(-\mathscr{L}(\mathcal{G}_2), B_4(S_2))$ are controllable. The nodes corresponding to sets S_1 and S_2 are half shaded in Fig. 1. Now, $B_3(S_1) \otimes B_4(S_2) = B_{12}(S)$ where $S = \{(1', 1), (1', 2)\}$, denoted by full shaded nodes in Fig. 1. Therefore, $(-\mathscr{L}(\mathcal{G}), B_{12}(S))$ is controllable.

We proceed to examine symmetry properties of p-simple composite graphs. The automorphism group of a graph describes its symmetries and has been previously shown to play an important role in the controllability of the pair $(-\mathscr{L}(\mathcal{G}), B(S))$ [13]. We will now extend this observation to more general graphs and matrix representations which are symmetry preserving.

Recall that the system matrix $A(\cdot)$ is symmetry preserving when $J\mathscr{A}(\mathcal{G}) = \mathscr{A}(\mathcal{G})J$ implies that $JA(\mathcal{G}) = A(\mathcal{G})J$ for all $\mathcal{G}$. As an aside, a special feature of $\mathscr{A}(\mathcal{G})$ is that simple eigenvalues are only present when $\mathrm{Aut}(\mathcal{G})$ has a specific structure.

Proposition IV.6: [42] If all eigenvalues of $\mathscr{A}(\mathcal{G})$ are simple, then every automorphism of $\mathcal{G}$ has order 1 or 2, i.e., $\sigma^2(i) = i$ for all $i \in V$.

A related property is that the assumption of eigenvalue distinctness in Theorem IV.1 boasts a special prime factor decomposition.

Proposition IV.7: For $A(\cdot) \in \mathbf{A}_\oplus$, if $A(\mathcal{G}_1 \square \mathcal{G}_2)$ is p-simple with respect to $A(\mathcal{G}_1)$ and $A(\mathcal{G}_2)$, then $\mathcal{G}_1$ and $\mathcal{G}_2$ are relatively prime, i.e., they contain no non-trivial prime factors in common.

Proof: The result follows from the eigenvalue properties of the Cartesian sum. ■

We now present observations that motivate the introduction of graph automorphisms for the analysis of composite networked systems.

Proposition IV.8: Assume that $A(\mathcal{G})$ is diagonalizable and symmetry preserving. Then the pair $(A(\mathcal{G}), B(S))$ is uncontrollable if $\mathcal{G}$ admits a nontrivial automorphism σ which fixes the input set S, i.e., $\sigma(i) = i$ for all $i \in S$.

Proof: Let the permutation matrix corresponding to the automorphism referenced in the statement of the proposition be $J \neq I$. Therefore,

$$JA(\mathcal{G}) = A(\mathcal{G})J$$

(as well as $J^T A(\mathcal{G}) = A(\mathcal{G})J^T$), and if the automorphism fixes the input set S, then

$$JB(S) = J^T B(S) = B(S)$$

since $JJ^T = J^T J = I$. Consider a left eigenvalue-eigenvector pair (λ, v) of $A(\mathcal{G})$; then as

$$(Jv)^T A(\mathcal{G}) = v^T A(\mathcal{G})J^T = \lambda v^T J^T = \lambda(Jv)^T$$

it follows that (λ, Jv) is also a left eigenvalue-eigenvector pair for $A(\mathcal{G})$ and $(v - Jv)^T A(\mathcal{G}) = \lambda(v - Jv)^T$. Since

$$\begin{aligned}(v - Jv)^T B(S) &= v^T B(S) - v^T J^T B(S)\\ &= v^T B(S) - v^T B(S) = 0\end{aligned}$$

and $v - Jv$ is nonzero for some v, as $A(\mathcal{G})$ is diagonalizable, by the PBH test the pair $(A(\mathcal{G}), B(S))$ is uncontrollable. ∎

We note that Proposition IV.8 is not sufficient for controllability as discussed in [13].

Proposition IV.8 highlights the requirement of selecting a set of inputs that break the symmetry structure of $\mathcal{G}$. Determining sets are a useful construct to describe this process.

Definition IV.9: A subset S of the vertices of a graph $\mathcal{G}$ is called a *determining set* if whenever $g, h \in \text{Aut}(\mathcal{G})$ so that $g(s) = h(s)$ for all $s \in S$, then $g = h$. The determining number of a graph $\mathcal{G}$, denoted $\text{Det}(\mathcal{G})$, is the smallest integer r so that $\mathcal{G}$ has a determining set of size r.

Another term for a determining set S is the *fixing set* due to the fact that no non-trivial automorphism of $\mathcal{G}$ fixes all members in S. Thus, a set $S \subseteq V(\mathcal{G})$ is a determining set if and only if the stabilizing set $\text{Stab}(S)$ of S, defined as

$$\text{Stab}(S) := \{g \in \text{Aut}(\mathcal{G}) \mid g(v) = v, \text{ for all } v \in S\}$$

only contains the trivial automorphism. Directly from Proposition IV.8 and the definition of determining sets we have the following corollary.

Corollary IV.10: A necessary condition for controllability of the pair $(A(\mathcal{G}), B(S))$ is that S is a determining set. Hence, for any controllable pair $(A(\mathcal{G}), B(S))$, one has $|S| \geq \text{Det}(\mathcal{G})$.

The automorphism group of a composite graph is intimately linked to the automorphism group of its prime factors. This link translates through the determining set of the composite graph summarized as follows.

Theorem IV.11: [43] Let $\mathcal{G} = \mathcal{G}_1^{k_1} \Box \cdots \Box \mathcal{G}_m^{k_m}$ be the prime factor decomposition for a connected graph $\mathcal{G}$. Then $\text{Det}(\mathcal{G}) = \max_i\{\text{Det}(\mathcal{G}_i^{k_i})\}$.

We now have the required ground work to state a consequence of the graph automorphism structure of the graph pertaining to Theorem IV.1.

Theorem IV.12: Under the assumption that $A(\mathcal{G}_1 \Box \mathcal{G}_2)$ is p-simple, consider the controllable networks $(A(\mathcal{G}_1), B(S_1))$ and $(A(\mathcal{G}_2), B(S_2))$, where $|S_1| = \text{Det}(\mathcal{G}_1)$ and $|S_2| = 1$. Then $S = S_1 \times S_2$ is the minimal cardinality set for which the pair $(A(\mathcal{G}_1 \Box \mathcal{G}_2), B(S))$ is controllable.

Proof: As $A(\mathcal{G}_1 \Box \mathcal{G}_2)$ is p-simple, from Proposition IV.7, neither $\mathcal{G}_1$ nor $\mathcal{G}_2$ contain the same nontrivial powers of prime graphs, i.e., $\mathcal{G}_1 = \mathcal{G}_{a1}^{k_{a1}} \Box \cdots \Box \mathcal{G}_{am}^{k_{am}}$ and $\mathcal{G}_2 = \mathcal{G}_{b1}^{k_{b1}} \Box \cdots \Box \mathcal{G}_{bn}^{k_{bn}}$ are prime factor decompositions, where $\mathcal{G}_{ai}$ is not isomorphic to $\mathcal{G}_{bj}$, for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Therefore, from Theorem IV.11, $\text{Det}(\mathcal{G}) = \max\{\text{Det}(\mathcal{G}_1), \text{Det}(\mathcal{G}_2)\}$. Moreover, as $(A(\mathcal{G}_2), B(S_2))$ is controllable from Theorem IV.1 it follows from Corollary IV.10 that

$$1 = |S_2| \geq \text{Det}(\mathcal{G}_2) \geq 1$$

and thereby $\text{Det}(\mathcal{G}) = \text{Det}(\mathcal{G}_1)$. Now $B(S_1) \otimes B(S_2) = B(S)$ for some $S \subseteq V(\mathcal{G}_1 \Box \mathcal{G}_2)$ and $|S| = |S_1||S_2| = |S_1|$. As $|S| = \text{Det}(\mathcal{G})$, it thus follows that the pair $(A(\mathcal{G}_1 \Box \mathcal{G}_2), B(S))$ is controllable with the smallest number of inputs. ∎

We now revisit Example IV.5 within the context of Theorem IV.12.

Example IV.13: Further examination of Example IV.5, we note that $\text{Aut}(\mathcal{G}_2) = \{\mathbf{id}, \sigma, \tau, \sigma\tau\}$, where $\mathbf{id}$ is the identity permutation, $\sigma(1, 2, 3, 4) = (1, 4, 3, 2)$ and $\tau(1, 2, 3, 4) = (3, 2, 1, 4)$. Hence, $\text{Det}(\mathcal{G}_2) = 2 = |S_2|$ and $|S_1| = 1$. Applying Theorem IV.12, this implies that S is the smallest controllable input set.

D. Layered Control

In this section, we examine a second network-centric compositional approach that allows for a direct analysis of the controllability properties of a composite network-of-networks *solely* in terms of the controllability of its factors.

This control scheme involves repeating the form of the control matrix B_1 to every $\mathcal{G}_1$ layer of the product $\mathcal{G} = \mathcal{G}_1 \Box \mathcal{G}_2$, motivating the terminology *layered control*. The corresponding composite input matrix is $B = B_1 \otimes I$, or with respect to Theorem IV.1, $B_2 = I$. Consequently, $B_2 \not\perp V$ for any V and so property (2) is trivially satisfied. Further, $(A(\mathcal{G}_2), B_2)$ is always controllable and so controllability of $(A(\mathcal{G}_1), B_1)$ is necessary and sufficient for controllability of $(A(\mathcal{G}), B)$.

As the Kronecker product exhibits permutation equivalency (see Section II-C), these results are equivalent to extending the control matrix B_2 to every $\mathcal{G}_2$ layer of $\mathcal{G}$ as well, producing a composite input matrix $B = I \otimes B_2$.

In fact under layered control assumption we can relax the diagonalizability requirement of Theorem IV.1, which sheds light on the controllability index, the gramian structure, and the structure of output feedback controllers.

1) Controllability Index and Diagonalizability for Layered Control: The other tool for establishing controllability that we will draw upon is the μ-controllability matrix defined for the pair (A, B) as

$$C_\mu(A,B) = [\, B \quad AB \quad A^2B \quad \cdots \quad A^{\mu-1}B \,]$$

when $C_n(A, B)$ has a full rank then (A, B) is controllable. Further, the smallest integer μ such that $C_\mu(A, B)$ has full rank is called the controllability index. The controllability index dictates the minimum degree required to achieve pole placement and model matching [6], and so serves as a useful measure of how effective the control is from a given input set.

The controllability index and the absence of the diagonalizability assumption for the layered control scheme is addressed in the following theorem.

Theorem IV.14: Consider $A(\cdot) \in \mathbf{A}_\oplus$ and graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ with n_1 and n_2 nodes, respectively. Then the pair $(A(\mathcal{G}_2), B_2)$ is controllable with the controllability index μ if and only if $(A(\mathcal{G}), B)$ is controllable with the controllability index μ, where $\mathcal{G} = \mathcal{G}_1 \square \mathcal{G}_2$ and $B = I_{n_1} \otimes B_2$.

Proof: For notational brevity, let $A_1 = A(\mathcal{G}_1)$, $A_2 = A(\mathcal{G}_2)$, and $A = A_1 \oplus A_2 = A(\mathcal{G}_1 \square \mathcal{G}_2)$. Define the μ-controllability matrix of the pair (A_2, B_2) as

$$\mathcal{C}_\mu(A_2,B_2) = [\, B_2 \quad A_2B_2 \quad A_2^2B_2 \quad \cdots \quad A_2^{\mu-1}B_2 \,].$$

If (A_2, B_2) is μ-controllable then $\operatorname{rank} \mathcal{C}_\mu(A_2, B_2) = n_2$. Now, $C_1(A, B) = I \otimes B_2$ and thus trivially $C_1(A, B)$ shares the same column space as $I \otimes B_2$. Let

$$\kappa_{p-1} := [\, I \otimes B_2 \quad I \otimes A_2B_2 \quad I \otimes A_2^2B_2 \cdots I \otimes A_2^{p-2}B_2 \,]$$

and assume that for positive integer p, $C_{p-1}(A, B)$ has the same column space as κ_{p-1}, denoted as $C_{p-1}(A, B) \sim \kappa_{p-1}$. Thereby

$$\begin{aligned} C_p(A,B) &= [\, C_{p-1}(A,B) \quad (A_1 \oplus A_2)^{p-1}(I \otimes B_2) \,] \\ &\sim [\, \kappa_{p-1} \quad (A_1 \oplus A_2)^{p-1}(I \otimes B_2) \,]. \end{aligned}$$

Further

$$\begin{aligned} &(A_1 \oplus A_2)^{p-1}(I \otimes B_2) \\ &\quad = \left[\sum_{k=0}^{p-1} \binom{p-1}{k} (A_1 \otimes I)^k (I \otimes A_2)^{p-1-k} \right] (I \otimes B_2) \\ &\quad = \sum_{k=1}^{p-1} \binom{p-1}{k} A_1^k \otimes A_2^{p-1-k} B_2 + I \otimes A_2^{p-1} B_2. \end{aligned}$$

We note that the column space of $M \otimes N$ for $M \in \mathbb{R}^{m \times m}$ lies in the column space of $I_m \otimes N$. Then $\sum_{k=1}^{p-1} \binom{p-1}{k} A_1^k \otimes A_2^{p-1-k} B_2$ lies in the column space of $\kappa_{p-1} \sim C_{p-1}(A, B)$ and so

$$C_p(A,B) \sim [\, C_{p-1}(A,B) \quad I \otimes A_2^{p-1}B_2 \,] \sim \kappa_p.$$

Hence, by induction, for all positive integers p, the matrix $\mathcal{C}_p(A, B)$ shares the same column space as κ_p. By permuting the columns of κ_p then

$$\begin{aligned} \kappa_p &\sim I_{n_1} \otimes [\, B_2 \quad A_2B_2 \quad A_2^2B_2 \quad \cdots \quad A_2^{p-1}B_2 \,] \\ &= I_{n_1} \otimes \mathcal{C}_p(A_2, B_2). \end{aligned}$$

Furthermore, as $\operatorname{rank} M \otimes N = \operatorname{rank} M \operatorname{rank} N$

$$\begin{aligned} \operatorname{rank} \mathcal{C}_p(A,B) &= \operatorname{rank} I_{n_1} \otimes \operatorname{rank} \mathcal{C}_p(A_2,B_2) \\ &= n_1 \operatorname{rank} \mathcal{C}_p(A_2,B_2). \end{aligned}$$

Note that $A_1 \oplus A_2 \in \mathbb{R}^{n_1n_2 \times n_1n_2}$ and (A, B) is controllable with controllability index μ if and only if $\operatorname{rank} \mathcal{C}_p(A_2, B_2) = n_2$ for $p \geq \mu$, thus proving the proposition. ■

An illustrative example of the layered control scheme is as follows.

Example IV.15: Consider the graph $\mathcal{G}_1$ and $\mathcal{G}_2$ from Fig. 1. For $S_1 = \{1'\}$ and $S_2 = \{1, 2\}$, the pairs $(-\mathscr{L}(\mathcal{G}_1), B_3(S_1))$ and $(-\mathscr{L}(\mathcal{G}_2), B_4(S_2))$ are controllable. The nodes corresponding to sets S_1 and S_2 are half shaded in Fig. 1. Now, $B_3(S_1) \otimes I = B_{12}(S_a)$, where $S_a = \{(1', 1), (1', 2), (1', 3), (1', 4)\}$, denoted by the lower half shaded nodes in Fig. 1. Similarly, $I \otimes B_4(S_2) = B_{12}(S_b)$ and $S_b = \{(1', 1), (1', 2), (2', 1), (2', 2), (3', 1), (3', 2)\}$, denoted by the upper half shaded nodes in Fig. 1. Therefore from Theorem IV-14, the pairs $(-\mathscr{L}(\mathcal{G}), B_{12}(S_a))$ and $(-\mathscr{L}(\mathcal{G}), B_{12}(S_b))$ are controllable.

Theorem IV.14 provides a useful tool of combining families of graphs with known controllability properties, such as the Laplacian of the path graphs, cycle graphs [44], and circulant graphs [41], with graphs where controllability is hard to establish, such as random graphs [45], [46], or graphs that require a large number of nodes to control, such as the complete graph.

Further, Theorem IV.14 can be combined with Theorem IV.12 to produce large scale controllable graphs. A composite graph $\mathcal{G}$ can be decomposed into $\mathcal{G}_a \square \mathcal{G}_b$, where $\mathcal{G}_a$ is the largest factor graph of $\mathcal{G}$ with simple eigenvalues and $\mathcal{G}_b$ has order n_b. In turn, $\mathcal{G}_a$ can be decomposed into its prime factors $\mathcal{G}_1 \square \ldots \square \mathcal{G}_k$ with the corresponding controllable matrix pairs $(A(\mathcal{G}_i), B_i)$ for $i = 1, \ldots, k$ and some $A(\cdot) \in \mathbf{A}_\oplus$. Therefore by Theorem IV.12, $(A(\mathcal{G}_a), B_1 \otimes \ldots \otimes B_k)$ is controllable and by Theorem IV.14, $(A(\mathcal{G}), B_1 \otimes \ldots \otimes B_k \otimes I_{n_b})$ is controllable. This technique is used in the following example to establish controllability of the grid $\mathcal{P}_2 \square \mathcal{P}_4 \square \mathcal{P}_5$.

Example IV.16: Denote the path graphs of length two, four, and five path graph as $\mathcal{P}_2, \mathcal{P}_4$, and $\mathcal{P}_5$, respectively. Since all path graphs are controllable from either end node, designating one of the ends as the first node, the pairs $(-\mathscr{L}(\mathcal{P}_4), B_4(S))$ and $(-\mathscr{L}(\mathcal{P}_5), B_5(S))$ are controllable for $S = \{1\}$. Noting that $\mathscr{L}(\mathcal{P}_4 \square \mathcal{P}_5)$ has distinct eigenvalues, and so is p-simple, from Theorem IV.12, the pair $(-\mathscr{L}(\mathcal{P}_4 \square \mathcal{P}_5), B_{20}(S))$ is controllable. Further applying Theorem IV.14 then it follows that $(-\mathscr{L}(\mathcal{P}_2 \square \mathcal{P}_4 \square \mathcal{P}_5), B_{40}(S'))$ for $S' = \{(1, 1), (2, 1)\}$ is controllable as $I_2 \otimes B_{20}(S) = B_{40}(S')$. Here we have a 40 node grid controllable from 2 nodes.

Interestingly, $\mathscr{L}(\mathcal{P}_2 \square \mathcal{P}_4 \square \mathcal{P}_5)$ has repeated eigenvalues and so, from the PBH test, at least two input nodes are required to form a controllable set. Hence, the set S' found in Example IV.16 is the smallest controllable input set.

2) Controllability Gramian for Layered Control: Computing the controllability Gramian of large scale Cartesian product composite networks is computationally expensive. The purpose of this section is to characterize the controllability Gramian of the product network in terms of the controllability Gramian of its factors for layered control.

Theorem IV.17: Let $\mathcal{G} = \mathcal{G}_1 \square \mathcal{G}_2$ and $B = I_{n_1} \otimes B_2$, where $A(\mathcal{G})$ is stable. Then, the controllability Gramian satisfies

$$P\left(A(\mathcal{G}), B\right) = W \mathbf{Diag}_i(P_i) W^T \tag{4}$$

where $W = I_{n_1} \otimes V$, $P_i = P(A(\mathcal{G}_2) + \mu_i I_{n_2}, B_2)$ and V is the orthogonal matrix of eigenvectors of $(1/2)(A(\mathcal{G}_1) + A(\mathcal{G}_1)^T)$ with associated eigenvalues $\mu_1, \ldots, \mu_{n_1}$. Moreover, the operator $\mathbf{Diag}_i(.)$ forms a block diagonal matrix consisting of its matrix arguments indexed by i.

Proof: First, we note that

$$\begin{aligned} &P\left(A(\mathcal{G}), B\right) \\ &= \int_0^\infty e^{A(\mathcal{G})t} \left(I_{n_1} \otimes B_2\right) \left(I_{n_1} \otimes B_2\right)^T e^{A(\mathcal{G})^T t} dt \\ &= \int_0^\infty e^{(A(\mathcal{G}_1) \oplus A(\mathcal{G}_2))t} \left(I_{n_1} \otimes B_2\right) \left(I_{n_1} \otimes B_2\right)^T \\ &\quad \times e^{(A(\mathcal{G}_1) \oplus A(\mathcal{G}_2))^T t} dt \\ &= \int_0^\infty \left(e^{A(\mathcal{G}_1)t} \otimes e^{A(\mathcal{G}_2)t}\right) \left(I_{n_1} \otimes B_2\right) \left(I_{n_1} \otimes B_2\right)^T \\ &\quad \times \left(e^{A(\mathcal{G}_1)^T t} \otimes e^{A(\mathcal{G}_2)^T t}\right) dt \\ &= \int_0^\infty e^{\left(A(\mathcal{G}_1) + A(\mathcal{G}_1)^T\right)t} \otimes \left(e^{A(\mathcal{G}_2)t} B_2 B_2^T e^{A(\mathcal{G}_2)^T t}\right) dt, \\ &= \int_0^\infty V \Sigma(t) V^T \otimes \left(e^{A(\mathcal{G}_2)t} B_2 B_2^T e^{A(\mathcal{G}_2)^T t}\right) dt \end{aligned}$$

where $e^{(A(\mathcal{G}_1) + A(\mathcal{G}_1)^T)t} = V \Sigma(t) V^T$ with $[\Sigma(t)]_{jj} = e^{2\mu_j t}$. It thus follows that:

$$\begin{aligned} &W^T P\left(A(\mathcal{G}), B\right) W \\ &= \int_0^\infty \mathbf{Diag}_i \left[e^{2\mu_i t} \left(e^{A(\mathcal{G}_2)t} B_2 B_2^T e^{A(\mathcal{G}_2)^T t}\right)\right] dt \\ &= \mathbf{Diag}_i \left[\int_0^\infty e^{2\mu_i t} \left(e^{A(\mathcal{G}_2)t} B_2 B_2^T e^{A(\mathcal{G}_2)^T t}\right) dt\right] \\ &= \mathbf{Diag}_i \left[\int_0^\infty e^{\left(A(\mathcal{G}_2) + \mu_i I_{n_2}\right)t} B_2 B_2^T e^{\left(A(\mathcal{G}_2)^T + \mu_i I_{n_2}\right)t} dt\right] \\ &= \mathbf{Diag}_i \left[P(A(\mathcal{G}_2) + \mu_i I_{n_2}, B_2)\right]. \end{aligned} \tag{5}$$

■

Let us now discuss some of the ramifications of (5) for generating large-scale controllable graphs. The identity (5) implies that, for example if $A(\mathcal{G}_2)$ is stable,

(a) when $A(\mathcal{G}_1)$ is negative semi-definite (semi-stable) with $\mu_i \leq 0$ for all i, we have $e^{2\mu_i t} \leq 1$, and therefore

$$P\left(A(\mathcal{G}_2) + \mu_i I_{n_2}, B_2\right) \preceq P\left(A(\mathcal{G}_2), B_2\right) \text{ for all } i \tag{6}$$

implying that the degree of controllability of the composite system $(A(\mathcal{G}), B)$ is bounded above in a p.s.d. ordering sense by the degree of controllability of $(A(\mathcal{G}_2), B_2)$.

(b) when $A(\mathcal{G}_2)$ is positive semi-definite with $\mu_i \geq 0$ for all i, we have $e^{2\mu_i t} \geq 1$, and therefore

$$P\left(A(\mathcal{G}_2) + \mu_i I_{n_2}, B_2\right) \succeq P\left(A(\mathcal{G}_2), B_2\right) \quad \text{for all } i \tag{7}$$

implying that the degree of controllability of the composite system $(A(\mathcal{G}), B)$ is in a sense bounded below by the degree of controllability of $(A(\mathcal{G}_2), B_2)$.

The relations in (5), (6) and (7) indicate that the controllability ellipsoid of the Cartesian product network with layered control has a nested structure. For every axis of the controllability ellipsoid of $(A(\mathcal{G}_2), B_2)$, there are n_1 corresponding axes of the controllability ellipsoid of $(A(\mathcal{G}), B)$. The eigenvalues of $A(\mathcal{G}_1)$ dictate the comparative scaling of the factor's axis and these n_1 composite axes.

The more stable each eigenvalue of the system matrix $A(\mathcal{G}_1)$, the shorter the corresponding n_1 composite axes, i.e., the less controllable the composite system. On the other hand, the more unstable $A(\mathcal{G}_1)$ the longer these n_1 axes, and so the more controllable the composite system. This observation also has direct ramifications for the H_2 norm of the composite network under this control structure.

Example IV.18: Define the matrix representation

$$[A(\mathcal{G})]_{ij} = \begin{cases} -w_{ji} & \text{for } i \neq j \\ -20 \sum_{i=1}^n w_{ii} + 11 \sum_{i \neq j} w_{ji} & \text{otherwise} \\ \quad - 10 \sum_{i \neq j} w_{ji} & \end{cases}.$$

In a more compact form $A(\mathcal{G}) = \mathscr{L}(\mathcal{G}) - 20\mathscr{D}_s(\mathcal{G}) + 10(\mathscr{D}_{in}(\mathcal{G}) - \mathscr{D}_{out}(\mathcal{G}))$. Hence, as $\mathbf{A}_\oplus$ is closed under addition then $A(\cdot) \in \mathbf{A}_\oplus$. Consider the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ depicted in Fig. 1. Then $A(\mathcal{G}_2) = \mathscr{L}(\mathcal{G}_2)$ and so has eigenvalues $0 = \mu_1 < \mu_2 \leq \mu_3 \leq \mu_4$ and

$$A(\mathcal{G}_1) = \begin{bmatrix} -15 & -1 & 0 \\ -1 & -6 & 0 \\ 0 & -1 & -8 \end{bmatrix}$$

which has all strictly negative eigenvalues. From (6) it follows that $\lambda_{min}(P(A(\mathcal{G})), B_1 \otimes I) = \lambda_{min}(P(A(\mathcal{G}_1), B_1))$. Therefore, the optimal single node input set S_1 in terms of the smallest controllability Gramian eigenvalue for the graph $\mathcal{G}_1$ generates the optimal set S for the graph $\mathcal{G}$ of the form $B_1(S_1) \square I_4$. The smallest eigenvalue of $P(A(\mathcal{G}_1), B_1)$ from single input nodes $1'$, $2'$, and $3'$ are 0.0218×10^{-5}, 0.3653×10^{-5}, and 0, respectively. Thus the optimal set in terms of the smallest controllability Gramian eigenvalue is $S_1 = \{2\}$ for $\mathcal{G}_1$ and $S = \{\{1, 2\}, \{2, 2\}, \{3, 2\}, \{4, 2\}\}$ for $\mathcal{G}$.

As mentioned in Section III, the discrete-time version of the above results follow analogously.

3) Output Feedback for Layered Control: One of the structural features of composite networks is that they exhibit repeated layers of the factors. In particular, Theorem IV.1 takes advantage of this feature by extending the controllable inputs in one factor to many. The same layering approach can be explored in the context of observable outputs. Our next proposition shows that the control input can similarly be designed for a factor and extended to the composite network-of-networks with the effect of generating distributed output feedback stabilization.

Proposition IV.19: Consider $A(\cdot) \in \mathbf{A}_{\oplus}$ and the n_1 node and n_2 node graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, where the matrix $A(\mathcal{G}_1)$ is semistable. If the dynamics $(A(\mathcal{G}_2), B_2, C_2)$ is stabilizable with output feedback $u_a = Ky_a$ for inputs u_a and outputs y_a, then the dynamics $(A(\mathcal{G}_1 \square \mathcal{G}_2), I_{n_1} \otimes B_2, I_{n_1} \otimes C_2)$ is stabilizable with the output feedback $u = (I_{n_1} \otimes K)y$ for inputs u and outputs y. Further, the control law can be realized with a local layer feedback across the layers of $\mathcal{G}_2$.

Proof: As K is stabilizing for the system described by the matrices $(A(\mathcal{G}_2), B_2, C_2)$, the matrix $A(\mathcal{G}_2) + B_2KC_2$ is stable. Consider the dynamics of the system $(A(\mathcal{G}_1 \square \mathcal{G}_2), I_{n_1} \otimes B_2, I_{n_1} \otimes C_2)$ with output feedback $u = (I_{n_1} \otimes K)y$. Then,

$$\begin{aligned}\dot{x}(t) &= \left(A(\mathcal{G}_1 \square\, \mathcal{G}_2) + (I_{n_1} \otimes B_2)\,(I_{n_1} \otimes K)\,(I_{n_1} \otimes C_2)\right) x(t)\\ &= \left(A(\mathcal{G}_1) \otimes I_{n_2} + I_{n_1} \otimes A(\mathcal{G}_2) + I_{n_1} \otimes B_2KC_2\right) x(t)\\ &= \left(A(\mathcal{G}_1) \oplus (A(\mathcal{G}_2) + B_2KC_2)\right) x(t).\end{aligned}$$

As the Cartesian sum of semistable and stable matrices is stable, $I_{n_1} \otimes K$ is stabilizing. Furthermore as $I_{n_1} \otimes K$ is block diagonal, the feedback loop can be broken into the inputs and outputs of each layer of $\mathcal{G}_2$. Specifically distributing the inputs and outputs accordingly, we have $u = [u_1^T, \ldots, u_{n_1}^T]^T$ and $y = [y_1^T, \ldots, y_{n_1}^T]^T$, where u_i and y_i are the inputs and outputs of the i-th layer of $\mathcal{G}_2$. Hence, the feedback for the composite network can be written in terms of the local layer feedback $u_i = Ky_i$, for $i = 1, \ldots, n_1$.

Remark IV.20: We note that Proposition IV.19 can be extended to the case where

$$\max \mathbf{Re}\{\lambda(A(\mathcal{G}_1))\} < -\mathbf{Re}\{\lambda_i(A(\mathcal{G}_2) + B_2KC_2)\} \text{ for all } i.$$

Proposition IV.9 describes a setup where we have a stabilizing distributed feedback on each factor layer requiring only local sensors and actuators placed on that layer. The following is an example illustrating this layered output feedback stabilization.

Example IV.21: Define the matrix representation

$$[A(\mathcal{G})]_{ij} = \begin{cases} w_{ji} & \text{for } i \neq j \\ \frac{1}{2}w_{ii} - \sum_{i \neq j} w_{ji} & \text{otherwise} \end{cases}$$

that can equivalently be represented as $A(\mathcal{G}) := -\mathscr{L}(\mathcal{G}) + (1/2)\mathscr{D}_s$; thus $A(\cdot) \in \mathbf{A}_{\oplus}$. For graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ described in Fig. 1, the matrix $A(\mathcal{G}_2) = -\mathscr{L}(\mathcal{G}_2)$ is semistable and

$$A(\mathcal{G}_1) = \begin{bmatrix} -\frac{1}{2} & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} \end{bmatrix}.$$

Consider the dynamics of the system described by the matrices $(A(\mathcal{G}_1), B(S_1), C(S_2))$, where $S_1 = \{1'\}$ and $S_2 = \{2'\}$; then the output feedback $u = ky$ is stabilizing for $k < -(1/2)$. Thus from Proposition IV.19, the output feedback $k \otimes I_4$ is stabilizing for the composite system, which is realized by the distributed feedback $u_{(1',i)} = ky_{(2',i)}$ for $i = 1, \ldots, 4$, where $u_{(1',i)}$ is the input applied to node $(1', i)$ and $y_{(2',i)}$ is the output measured from node $(2', i)$.

V. Filtering on Composite Social Networks

In this section we examine the problem of estimating the internal state of a social influence network utilizing the observability properties of the Cartesian product networks. The underlying models have been adopted from [29], [30]. The time invariant model described in [29], and the time varying model in [30], postulate recursive formulas for the influence process in a group of n agents. For the time invariant case, the recursion assumes the form

$$x(t+1) = WF(\mathcal{G})x(t) + (I - W)x(0), \quad t = 1, 2, \ldots \tag{8}$$

where $x(0)$ and $x(t)$ denote, respectively, the n dimensional vectors of agents' opinions at the initial time and time t on an issue; the interpersonal influences are captured by the matrix $F(\mathcal{G}) \in \mathbb{R}^{n \times n}$, where $0 \leq F_{ij} \leq 1, \sum_j^n F_{ij} = 1$, and $\mathcal{G}$ is the underlying influence network. In (8), the diagonal matrix $W = \operatorname{diag}(w_{11}, w_{22}, \cdots, w_{nn})$, with $0 \leq w_{ij} \leq 1$, captures the agents' susceptibilities to interpersonal influence on the issue.

A Cartesian network structure is not uncommon in social networks due to the categorization of the population into common interest groups, forming a layered structure of social interactions. For example consider Padgett's study of elite family structure in fifteenth-century Florence [47]. Supporting Padgett's analysis includes more than 12,000 elections, 3,000 business partnerships, 14,000 loans, and 10,000 marriages, providing a natural grouping of the 15 elite families into their political, business, financial, and social members. The evidence strongly suggests that families, designated by common surnames, were the naturally-existing interest groups that tend to unify internally in the face of political crisis. The Florentines, for example, were particularly aware of the importance of the family unit, and marriages were arranged for political, social, business and financial gain.

Interactions among families involved interactions on the political platform between political members of the families and similarly business, financial and social members. Due to the size of families, within a given family, these four core member groups would interact in a similar fashion as within other large families. This network can be realized through a Cartesian product of the family members interaction graph and inter-family graph. The family member graph $\mathcal{G}_2$ with nodes a, b, c and d, corresponds to the network amongst its political, business, financial and social members, respectively, as depicted through the corresponding matrix $A(\mathcal{G}_2)$ in Fig. 2(b). Similarly, the family interaction graph $\mathcal{G}_1$ is the famous Florentine family graph [47], and is depicted in Fig. 2(a) with nodes representing the elite fifteen families of Florence in the 15th century. The composite graph $\mathcal{G}_1 \square \mathcal{G}_2$ corresponds to the resulting 60 member group social interactions. The discrete time opinion dynamics of the interaction have a state matrix $A(\mathcal{G}) = \exp(-\delta(\mathscr{L}(\mathcal{G}) + \max(\mathscr{D}_s(\mathcal{G}))I))$ and $\mathbf{A}_{\otimes}$ is

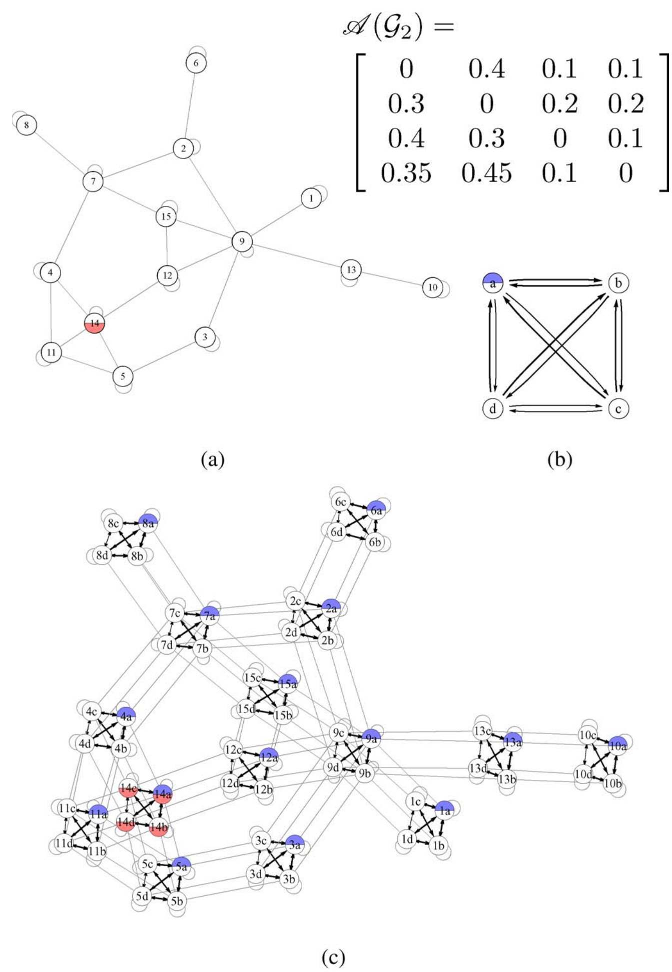

Fig. 2. Factor graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ and composite graph $\mathcal{G}_1 \square \mathcal{G}_2$. For image clarity all edge weights in $\mathcal{G}_1$ are 1, in $\mathcal{G}_2$ and $\mathcal{G}_1 \square \mathcal{G}_2$ the edge weights can be divulged through the definition of $\mathscr{A}(\mathcal{G}_2)$ and $\mathcal{G}_1$. (a) $\mathcal{G}_1$; (b) $\mathcal{G}_2$ and its adjacency matrix $\mathscr{A}(\mathcal{G}_2)$; (c) $\mathcal{G} = \mathcal{G}_1 \square \mathcal{G}_2$.

closed under multiplication $A(\cdot) \in \mathbf{A}_{\otimes}$.[3] This is an adaptation of the traditional opinion dynamics in (8) when $W = I$ as $\exp(-\delta \mathscr{L}(\mathcal{G}))$ is of the form $F(\mathcal{G})$, with the exception of the self loop matrix term, which increases the influence of an individual's opinion on its subsequent time evolution.

It is often unrealistic or expensive to survey the records of all groups in the population. An alternative is to sample the network, and subsequently estimate its state dynamics through an *opinion dynamics filter*. However, a requirement for designing such an estimator is observability of the social network. In this venue, if S is the output nodes of the graph, the discrete social interaction dynamics assumes the form

$$x(k+1) = A(\mathcal{G}_1 \square \, \mathcal{G}_2) x(k), \quad y(k) = C_{60}(S) x(k) \qquad (9)$$

where $t = k\delta$, for some time step $\delta = 0.05$ year.

Assuming that all members of one demographic in the social network can be observed, e.g., all business dealings, then the dual statement of Theorem IV.17 for discrete time systems provides the necessary conditions for the filter design. Specifically, the dynamics are observable and, under one demographic, in fact optimal for $C_{60}(S) = I \otimes C_4(S_2)$ (requiring minimal number of outputs), where $S_2 = \{a\}$ and $S = \{1a, 2a, \ldots, 15a\}$. This information would be attractive to historians, as the opinions of all 60 member groups of the social network can be efficiently divulged by surveying the political data in the society. If instead an historian was interested in the best family to survey, then the dual statement of Theorem IV.17 for discrete time systems can be applied to $\mathcal{G}_1$, leading to an observable dynamics with $C_{60}(S) = C_{15}(S_1) \otimes I$, where $S_1 = \{14\}$ and $S = \{14a, 14b, 14c, 14d\}$, i.e., every member group of the family 14 is observed directly. As $A(\mathcal{G}_1 \square \mathcal{G}_2)$ has simple eigenvalues (and so is p-simple), using Theorem IV.1, the dynamics with the observation matrix $C_{60}(S) = C_{15}(S_1) \otimes C_4(S_2)$ is observable, where $S = \{14a\}$. Therefore, in this scenario, surveying the political data of family 14 would provide a mechanism for estimating the opinion of all members of the social network.

A discrete Kalman filter[4] was applied to the social dynamics (9) with $\mathcal{G}_1$ and $\mathcal{G}_2$ as depicted in Fig. 2 where $S = \{1a, 2a, \ldots, 15a\}$, i.e., all families political groups in the social network are observed. The resultant mean squared error and a sample state estimate over time is provided in Fig. 3, supporting the observability of the pair $(A(\mathcal{G}_1 \square \mathcal{G}_2), C_{60}(S))$.

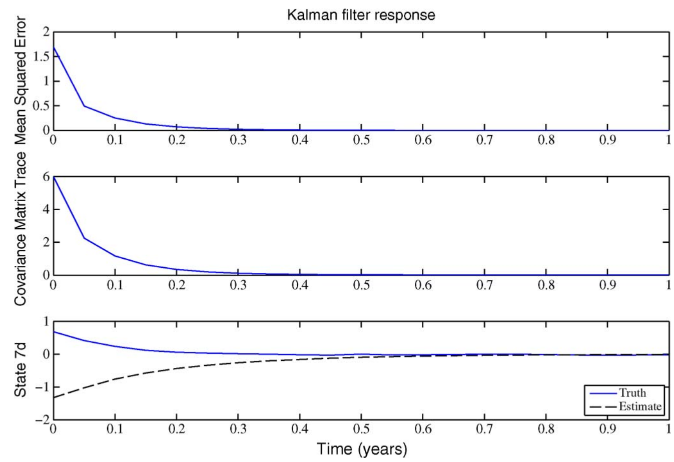

Fig. 3. The mean squared error, the trace of the covariance matrix and state of a random node (node 7d) over time for the discrete Kalman filter pertaining to Fig. 2.

VI. Conclusion

This paper presents an analysis of the controllability and observability of networks-of-networks formed by the graph Cartesian product of its factors. We provided insights into the effectiveness of two composite control schemes using graph symmetry and the controllability Gramian. Future work of particular interest involves extending these results to other types of graph products such as the direct product. Another extension of interest is controllability features of composite networks that in some suitable sense, approximate a Cartesian product of small factor networks.

[3] The matrix representations $A_1(\mathcal{G}) = \exp(-\delta \mathscr{L}(\mathcal{G}))$ and $A_2(\mathcal{G}) = \exp(-\delta \max(\mathscr{D}_s(\mathcal{G}))I)$ are both in $\mathbf{A}_{\otimes}$ and $A(\mathcal{G}) = A_1(\mathcal{G})A_2(\mathcal{G})$ as $\mathscr{L}(\mathcal{G})$ and $\max(\mathscr{D}_s(\mathcal{G}))I$ commute [48].

[4] For a detailed description of the discrete Kalman filter see [49].

APPENDIX

Proposition A.1: The matrix representation (2) exhibits the property $A(\mathcal{G}_1 \square \mathcal{G}_2) = A(\mathcal{G}_1) \oplus A(\mathcal{G}_2)$.

Proof: Examining the diagonal entries of $A(\mathcal{G}_1 \square \mathcal{G}_2)$, and noting that $w_{(i,p),(i,p)} = \delta_{pp} w_{ii} + \delta_{ii} w_{pp} = w_{ii} + w_{pp}$, $w_{(j,q),(i,p)} = \delta_{pq} w_{ji} + \delta_{ij} w_{qp}$, and $f(0, w_{ji}) = 0$ then

$$\begin{aligned}
&[A(\mathcal{G}_1 \square \mathcal{G}_2)]_{(i,p),(i,p)} \\
&= r w_{(i,p),(i,p)} + \sum_{(j,q)\neq(i,p)} f\left(w_{(i,p),(j,q)}, w_{(j,q),(i,p)}\right) \\
&= r w_{(i,p),(i,p)} + \sum_{j\neq i} f\left(w_{(i,p),(j,p)}, w_{(j,p),(i,p)}\right) \\
&\quad + \sum_{p\neq q} f\left(w_{(i,p),(i,q)}, w_{(i,q),(i,p)}\right) \\
&= r\left(w_{ii} + w_{pp}\right) + \sum_{j\neq i} f\left(\delta_{pp} w_{ij} + \delta_{ji} w_{pp}, \delta_{pp} w_{ji} + \delta_{ij} w_{pp}\right) \\
&\quad + \sum_{p\neq q} f\left(\delta_{qp} w_{ii} + \delta_{ii} w_{pq}, \delta_{pq} w_{ii} + \delta_{ii} w_{qp}\right) \\
&= r w_{ii} + \sum_{j\neq i} f(w_{ij}, w_{ji}) + r w_{pp} + \sum_{p\neq q} f(w_{pq}, w_{qp}) \\
&= [A(\mathcal{G}_1)]_{ii} + [A(\mathcal{G}_2)]_{pp} \\
&= [A(\mathcal{G}_1) \oplus A(\mathcal{G}_2)]_{(i,p),(i,p)}.
\end{aligned}$$

Examining the remaining elements of $A(\mathcal{G}_1 \square \mathcal{G}_2)$, and noting that $g(0, w_{ji}) = 0$, it follows that:

$$\begin{aligned}
[A(\mathcal{G}_1 \square \mathcal{G}_2)]_{(i,p),(j,q)} &= g\left(w_{(i,p),(j,q)}, w_{(j,q),(i,p)}\right) \\
&= g(\delta_{qp} w_{ij} + \delta_{ji} w_{pq}, \delta_{pq} w_{ji} + \delta_{ij} w_{qp}) \\
&= \delta_{pq} g(w_{ij}, w_{ji}) + \delta_{ij} g(w_{pq}, w_{qp}) \\
&= \delta_{pq} [A(\mathcal{G}_1)]_{ij} + \delta_{ij} [A(\mathcal{G}_2)]_{pq} \\
&= [A(\mathcal{G}_1) \oplus A(\mathcal{G}_2)]_{(i,p),(j,q)}.
\end{aligned}$$

■

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