Problem 1

For simulation, we assume the following initial conditions:

\[ R_0 = 300 + 6378 \text{ km}, \quad \vec{r} = [1.3, 0.8, -1.2] \text{ km}, \quad \mu = 398600.4, \quad n = \sqrt{\frac{\mu}{R_0^3}} = 0.0010831, \]

Rendezvous time \( t = 100 \text{ s}, \quad \vec{\dot{r}} = [\dot{x}_0^+, \dot{y}_0^+, \dot{z}_0^+] = [0.0309, -0.0023, 0.0001] \]

From the solution of the relative motion in the circular orbit, we obtain the required velocity for rendezvous as

\[ \dot{x}_0^+ = -0.0122 \text{ km/s}, \quad \dot{y}_0^+ = -0.0095 \text{ km/s}, \quad \dot{z}_0^+ = 0.0119 \text{ km/s} \]  \( \text{(1)} \)

and the velocity at the point of rendezvous yields

\[ \dot{x}(100s) = -0.0137 \text{ km/s}, \quad \dot{x}(100s) = -0.0066 \text{ km/s}, \quad \dot{x}(100s) = 0.0120 \text{ km/s}. \]  \( \text{(2)} \)

Thus, the impulse to boost up to the required velocity and the second impulse to cancel out the residual velocity are given as

\[ \triangle v_1 = 0.0453 \text{ km/s}, \quad \triangle v_2 = 0.0194 \text{ km/s} \]  \( \text{(3)} \)
\[ r_{\dot{p}}(3) = -r(3) \frac{n}{\tan(n t)}; \]

% velocity at time t
\[ r_{\dot{t}}(1) = 3n \sin(n t) r(1) + \cos(n t) r_{\dot{p}}(1) + 2 \sin(n t) \]
\[ \frac{r_{\dot{p}}(1)}{} + \frac{r_{\dot{p}}(1)}{} + \frac{r_{\dot{p}}(1)}{}; \]
\[ r_{\dot{t}}(2) = 6n(-1-\cos(n t)) r(1) - 2 \sin(n t) r_{\dot{p}}(1) \]
\[ + 3 \cos(n t) r_{\dot{p}}(2); \]
\[ r_{\dot{t}}(3) = -n \sin(n t) r(3) + \cos(n t) r_{\dot{p}}(3); \]

% two impulses
\[ \text{del}_v_1 = \|r_{\dot{p}} - r_{\dot{t}}\| \]
\[ \text{del}_v_2 = \|r_{\dot{t}}\| \]

%------------------------------------------------
\[ t_{\text{span}} = 0:1:t; \]
\[ r_{t} = r; \]

for i=2:length(t_{\text{span}})
\[ r_{t}(i,1) = (4-3 \cos(n t_{\text{span}}(i))) r(1) + \left( \sin(n t_{\text{span}}(i))/n \right) \]
\[ + 2(1-\cos(n t_{\text{span}}(i)))/n r_{\dot{p}}(2); \]
\[ r_{t}(i,2) = (6 \sin(n t_{\text{span}}(i)) - 6 n t_{\text{span}}(i)) r(1) + r(2) \]
\[ - 2(-1-\cos(n t_{\text{span}}(i)))/n r_{\dot{p}}(1) + (4 \sin(n t_{\text{span}}(i)) \]
\[ /n - 3 t_{\text{span}}(i)) r_{\dot{p}}(2); \]
\[ r_{t}(i,3) = \cos(n t_{\text{span}}(i)) r(3) + \sin(n t_{\text{span}}(i))/n r_{\dot{p}}(3); \]

% rotation matrix about 3-axis
\[ C_3 = \begin{bmatrix} \cos(n t_{\text{span}}(i)) & \sin(n t_{\text{span}}(i)) & 0 \\ -\sin(n t_{\text{span}}(i)) & \cos(n t_{\text{span}}(i)) & 0 \\ 0 & 0 & 1 \end{bmatrix} \]
\[ \text{R}_i(i,:) = (C_3 r_{t}(i,:))'; \]
\[ \text{R}_i2(i,:) = \begin{bmatrix} \text{R}_0 \cos(n t_{\text{span}}(i)) \\ \text{R}_0 \sin(n t_{\text{span}}(i)) \\ 0 \end{bmatrix}; \]
\]
\[ \text{end} \]
\[ \text{figure}(1); \text{plot}(t_{\text{span}},r_{t},'.-'); \]
\[ \text{xlabel}('time'); \text{ylabel}('distance'); \text{legend}('x','y','z') \]
\[ \text{figure}(2); \text{plot}(\text{R}_i(:,1),\text{R}_i(:,2),'.-',\text{R}_i2(:,1),\text{R}_i2(:,2),'.-'); \]
\[ \text{xlabel}('x distance'); \text{ylabel}('y distance'); \text{legend}('Target','Chase') \]

Problem 2

(a) First, we know that \( n = 0.0011 \) since the target spacecraft is in 400km circular orbit. From an elliptic formation flying orbit equation, we can find initial conditions at time \( t \) resulting in the formation flying orbit as

\[ \frac{\dot{y}_t}{n} = 0 \text{ km/s}, \quad \dot{x}_t = 0 \text{ km}, \quad \text{and} \quad x^2 + \left( \frac{y - y_t + \frac{2 \pi i}{n}}{\frac{4}{n}} \right)^2 = \left( \frac{\dot{x}_t}{n} \right)^2 \tag{1} \]

Given the shape of the orbit trajectory (ellipse equation), we can find

\[ \left( \frac{\dot{x}_t}{n} \right)^2 = b^2 \rightarrow \dot{x}_t = nb = 0.00028285 \text{ km/s} \tag{2} \]
\[ -y_t + \frac{2 \dot{x}_t}{n} = 0 \rightarrow y_t = 0.5 \text{ km} \tag{3} \]

Thus, any spacecraft having such initial conditions will be flying an elliptic formation flying orbit around the target. Since we need 6 spacecraft flying uniformly in the orbit, we can let each spacecraft have such initial conditions.
$T/6$ interval as

\[
\begin{align*}
\text{S/C 1: } & \quad x_t = 0, \quad \dot{x}_t = 0.00028285, \quad y_t = 0.5, \quad \dot{y}_t = 0, \quad \text{at } t = 925.58s \\
\text{S/C 2: } & \quad x_t = 0, \quad \dot{x}_t = 0.00028285, \quad y_t = 0.5, \quad \dot{y}_t = 0, \quad \text{at } t = 1851.2s \\
\text{S/C 3: } & \quad x_t = 0, \quad \dot{x}_t = 0.00028285, \quad y_t = 0.5, \quad \dot{y}_t = 0, \quad \text{at } t = 2776.8s \\
\text{S/C 4: } & \quad x_t = 0, \quad \dot{x}_t = 0.00028285, \quad y_t = 0.5, \quad \dot{y}_t = 0, \quad \text{at } t = 3702.4s \\
\text{S/C 5: } & \quad x_t = 0, \quad \dot{x}_t = 0.00028285, \quad y_t = 0.5, \quad \dot{y}_t = 0, \quad \text{at } t = 4628s \\
\text{S/C 6: } & \quad x_t = 0, \quad \dot{x}_t = 0.00028285, \quad y_t = 0.5, \quad \dot{y}_t = 0, \quad \text{at } t = 5553.6s,
\end{align*}
\]

where $T = 2\pi/n = 5553.5s$ is an orbit period.

We assume that 6 spacecraft have all initial conditions are zeros, we can find two impulse maneuvers for each spacecraft. For example, for the first spacecraft, the required velocity(after burn) is

\[
\dot{x}_0^+ = -0.00044219, \quad \dot{y}_0^+ = 0.00038295,
\]

and the velocity at a given time $t$,

\[
\dot{x}_t^- = 0.00044219, \quad \dot{y}_t^- = 0.00038295
\]

Thus, the impulse to boost up to the required velocity and the second impulse to match the orbit velocity are given as

\[
\Delta v_1 = 0.00058497, \quad \Delta v_2 = 0.00041478
\]

The trajectory of the spacecraft 1 with respect to the orbit frame(target) is depicted in the Fig. 2 for 5000s. Note that the orbit period is 5535s.

Figure 2: Formation flying orbit trajectory with respect to the orbit frame.

```matlab
% Pb2, HW7 for spacecraft #1
clear all;clc;close all;syms dx_0 dy_0
R0 = 6378 + 400;mu = 398600.4;t = 925.58;
n = sqrt(mu/R0^3);dx_t_p= 0.00028285;dy_t_p = 0;
x_0 = 0;y_0 = 0;x_t = 0;y_t = 0.5;
% required velocity to get to x,y at t
```
result = solve(x_t == (4-3*cos(n*t))*x_0 + sin(n*t)/n*dx_0 +... 
    2*(1-cos(n*t))/n*dy_0,y_t == (6*sin(n*t)-6*n*t)*x_0 + y_0 +...
    2*(-1+cos(n*t))/n*dx_0 + (4*sin(n*t)/n-3*t)*dy_0); 
 dx_0_p = double(result.dx_0) 
 dy_0_p = double(result.dy_0) 

 vel at time t 
 dx_t_m = 3*n*sin(n*t)*x_0 + cos(n*t)*dx_0_p + 2*sin(n*t)*dy_0_p 
 dy_t_m = 6*n*(-1+cos(n*t))*x_0 - 2*sin(n*t)*dx_0_p + (-3+4*cos(n*t))*dy_0_p

 % two impulses 
 del_v_1 = sqrt(dx_0_p^2 + dy_0_p^2) 
 del_v_2 = sqrt((dx_t_m-dx_t_p)^2 + (dy_t_m-dy_t_p)^2) 

 tspan = 1:1:5000; r = [x_0,y_0]; 
 for i = 1:925 
    r_t(i,1) = (4-3*cos(n*tspan(i)))*r(1) + (sin(n*tspan(i))/n)... 
              *dx_0_p + (2*(1-cos(n*tspan(i)))/n)*dy_0_p; 
    r_t(i,2) = (6*sin(n*tspan(i))-6*n*tspan(i))*r(1) + r(2)... 
              +2*(-1+cos(n*tspan(i)))/n*dx_0_p + (4*sin(n*tspan(i))... 
              /n - 3*tspan(i))*dy_0_p 
    r_t(i,3) = 0; 
 end 

 r = [x_t,y_t]; 
 for i = 1:tspan(end) 
    r_t(925+i,1) = (4-3*cos(n*tspan(i)))*r(1) + (sin(n*tspan(i))/n)... 
                   *dx_t_p + (2*(1-cos(n*tspan(i)))/n)*dy_t_p; 
    r_t(925+i,2) = (6*sin(n*tspan(i))-6*n*tspan(i))*r(1) + r(2)... 
                   +2*(-1+cos(n*tspan(i)))/n*dx_t_p + (4*sin(n*tspan(i))... 
                   /n - 3*tspan(i))*dy_t_p 
    r_t(925+i,3) = 0; 
 end 

 figure(2);plot(r_t(:,2),r_t(:,1),'.-');axis equal;legend('S/C 1') 
xlabel('y distance');ylabel('x distance');axis([-0.55 0.55 -0.3 0.3])
Problem 3

Let the Earth’s radius be \( r_e = 6378.1 \) km, the altitude is \( h = 300 \) km, the circular orbit radius of the Earth about the sun is \( r_{\text{E}/\text{S}} = 149.6 \cdot 10^6 \) km, while the Earth’s gravitational constant is \( \mu_{\text{Earth}} = 3.986 \cdot 10^7 \) km³/s² and the sun gravity constant is \( \mu_\odot = 1.326 \cdot 10^{11} \) km³/s².

a) The orbit radius of the satellite relative to the Earth center is

\[
r_s = r_e + h = 6678.1 \text{ km}
\]

The Earth’s gravitational acceleration magnitude at this altitude is given by

\[
a_{\text{Earth}} = \frac{\mu_{\text{Earth}}}{r_s^2} = 0.00893782 \text{ km/s}^2
\]

b) The acceleration magnitude of the Earth due to the Sun’s gravity field is given by

\[
a_{\text{Earth}/\odot} = f_{\text{Earth}/\odot} = \frac{G\mu_{\odot}m_s}{r_{\text{Earth}/\odot}^2} = \frac{\mu_\odot}{r_{\text{Earth}/\odot}^2}
\]

The acceleration magnitude of the satellite due to the Sun’s gravity can be approximated using \( r_{\text{s}/\odot} \approx r_{\text{E}/\odot} \):

\[
a_{\text{s}/\odot} = a_{\text{Earth}/\odot} = \frac{f_{\text{s}/\odot}}{m_s} = \frac{G\mu_{\odot}m_s}{r_{\text{s}/\odot}^2} \approx \frac{\mu_\odot}{r_{\text{Earth}/\odot}^2} = a_{\text{Earth}/\odot}
\]

To estimate the angle \( \delta \) between the Earth/Sun radius shown in Figure 1 of the problem statement and the Satellite/Sun radius shown, we find

\[
\sin \delta \approx \frac{r_{\text{s}/\odot}}{r_{\odot}/\odot}
\]

The desired disturbance acceleration magnitude \( a_d \) is computed as the difference between the sun influence on the satellite and the Earth (see Eq. (9.43)), which can be approximated as

\[
a_d \approx a_{\text{Earth}/\odot} \sin \delta = a_{\text{Earth}/\odot} \frac{r_{\text{s}/\odot}}{r_{\text{Earth}/\odot}} = \frac{\mu_\odot}{r_{\text{Earth}/\odot}^2} \frac{r_{\text{s}/\odot}}{r_{\text{Earth}/\odot}} = \mu_\odot \frac{r_{\text{s}/\odot}}{r_{\odot}/\odot} = 2.64485 \cdot 10^{-10} \text{ km/s}^2
\]

c) Note that this sun perturbation on the relative motion of the satellite motion about the Earth is about 7 orders of magnitudes smaller than the Earth’s gravitational influence on the satellite.

Problem 4

a) Taking the magnitude of the inertial position and velocity vectors, we find

\[
\frac{1}{a} = \frac{2}{\tau} = \frac{v^2}{\mu_{\text{Earth}}} = 0.0001492537313 \text{ km}^{-1}
\]

Since \( 1/a > 0 \), the orbit must be elliptical! A parabolic orbit would have \( 1/a = 0 \), while a hyperbolic orbit would have \( 1/a < 0 \). Thus, the semi-major axis is given by

\[
a = 6700.0000000 \text{ km}
\]
b) The angular momentum vector $\mathbf{h}$ is given by

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} = \begin{pmatrix} 22654.61356308633 \\ -26998.71710264165 \\ 37794.90823508838 \end{pmatrix} \text{ km}^2/\text{s}$$

The eccentricity vector is then given by

$$\mathbf{c} = \mathbf{v} \times \mathbf{h} - \frac{\mu c}{r} = \begin{pmatrix} 142.6452540483151 \\ 337.9676231756457 \\ 155.9236261202605 \end{pmatrix} \text{ km}^3/\text{s}^2$$

The magnitude of the eccentricity vector is

$$c = \sqrt{\mathbf{c} \cdot \mathbf{c}} = 398.5999999999929 \text{ km}^3/\text{s}^2$$

The eccentricity is then given by

$$e = \frac{c}{\mu} = 0.001$$

c) The periapses radius is $r_p = a(1 - e) = 6693.3 \text{ km}$. Since this is greater than the Earth’s radius, the satellite will not hit the Earth.

**Problem 5**

a) Given $\mu = 0.00095369$, we can obtain the Lagrangian points for Sun-Jupiter system as

$$L_1 = (-1.06886, 0), \quad L_2 = (-0.93246, 0), \quad L_3 = (1.00041, 0), \quad L_{4,5} = (-0.499046, \pm 0.8660254)$$

(1)

b) We can find the non-dimensional natural frequencies by investigating the eigenvalue of the matrix from the form of $\dot{x} = Ax$. For the Sun-Jupiter system, we have $\mu = 0.00095369$ and

$$\lambda_{1,2} = \pm 0.0805, \quad \lambda_{3,4} = \pm 0.9968, \quad \lambda_{5,6} = \pm i.$$  

Therefore, we have non-dimensional natural frequencies:

$$0.0805, 0.9968, \text{ and } 1$$

(3)

c) We can simulate the trajectory by integrating CR3BP differential equations. From the simulation result (Fig. 3), we obtain the next closest pass time of the comet by Jupiter as

$$t = 9.003 = \frac{T}{2\pi} = 1.435T,$$

(4)

where $T$ denotes the period of the Sun-Jupiter system.

% Problem 5c, HW#7
function pb5b
clcs;global rho
X0=[-1;0;0;1.5];
rho=0.00095369; %nondimensional mass of second primary
options=odeset('MaxStep',0.001,'RelTol',1e-5);
[T,Y]=ode45(@cr3b,[0,3*pi],X0,options);
for i=1:length(Y)
    dist(i) = sqrt((Y(i,1)+1-rho)^2 + Y(i,2)^2);
end

end
d) The zero-relative-velocity surface contour can be found from Eq. 10.95. If we plot the surface near L1, L2, and L4 for the Sun-Jupiter system, we obtain the following figures.
function pb5d

close all; clc;

global rho

rho=0.00096369;

% for L2, L1
x=-1.2:0.05:-0.4; y=-0.5:0.05:0.5;

figure(1); L2 = [-0.93246,0]; L1=[-1.06886,0]; L3=[1.00041,0]; L4=[-0.499046,0.8660254];

[X,Y,C]=arry(x,y); surf(X,Y,C)

hold on; plot3(L2(2),L2(1),-2.6,'ro',L1(2),L1(1),-2.6,'ro')

text(L2(2),L2(1),-2.6, L1 , 'FontSize', 12)

axis square; xlabel('y'); ylabel('x')

% for L4
figure(2); x=-1:0.05:1; y=0.5:0.05:1; clear X Y C

[X,Y,C]=arry(x,y);

surf(X,Y,C)

hold on; plot3(L4(2),L4(1),-2.2,'ro');

text(L4(2),L4(1),-2.2, L4 , 'FontSize', 12)

axis square; xlabel('y'); ylabel('x')

function [X,Y,C] = arry(x,y)

global rho

for i=1:length(x)
    for j=1:length(y)
        X(i,:)=x(i); Y(:,j)=y(j);
        r1=sqrt((x(i)-rho)^2+y(j)^2);
        r2=sqrt((x(i)+1-rho)^2+y(j)^2);
        C(i,j)=-0.5*(x(i)^2+y(j)^2)-(1-rho)/r1-rho/r2;
    end
end