Problem 1. You are flying in a lunar transit spacecraft and your current attitude relative to an inertial frame is given in terms of 3-2-1 Euler angles as (230, 70, and 103 deg). You are interested in docking onto the space station, which attitude relative to the same inertial frame is given through the Euler parameters (unit quaternion) \((\beta_0, \beta_1, \beta_2, \beta_3) = (0.328474, -0.437966, 0.801059, -0.242062)\). What is the attitude of your spacecraft relative to the space station? Express your answer using the principal rotation angle \(\Phi\) and principal rotation axis \(\hat{e}\).

Problem 2. The initial 3-2-1 Euler angles yaw, pitch, and roll of a vehicle are \((\psi_0, \theta_0, \phi_0) = (40, 30, 80)\) deg at time \(t_0\). Assume that the angular velocity vector of the craft is given in body frame components as

\[
\vec{\omega} = \begin{bmatrix} \sin(0.1t) \\ 0.01 \\ \cos(0.1t) \end{bmatrix} \text{ 20deg/s,} \tag{1}
\]

where the time \(t\) is assumed to be given here in units of seconds.

(a) Translate this initial attitude description into the corresponding Euler parameters.

(b) Write a program to numerically integrate the Euler parameters (unit quaternion) over a simulation time of 1 min. Note that you must integrate these equations using radians as the angular units. Plot the four Euler parameter time histories.

(c) Given the result of the numerical integration, plot the Euler parameter constraint \(\beta_0^2 + \beta_1^2 + \beta_2^2 + \beta_3^2\); comment on these values.

Problem 3. The spectral decomposition is a similarity transformation which converts a similar matrix to be diagonal as

\[
^B[I_c]_{\text{nondiag}} = C^F[I_c]_{\text{diag}}C^{-1}, \tag{2}
\]

where

\[
^F[I_c]_{\text{diag}} = \text{Diag}[\lambda_1, \lambda_2, \lambda_3], \quad C = [v_1, v_2, v_3] \tag{3}
\]

and \(v_1, v_2, v_3\) denote unitized eigen-vectors associated with \(\lambda_1, \lambda_2, \lambda_3\) of \(^B[I_c]_{\text{nondiag}}\).

(a) Calculate the principal inertia matrix \(^F[I_c]\), and the principal rotation matrix \([BF]\), for a spacecraft with the following inertia matrix:

\[
^B[I_c] = \begin{bmatrix} 12500 & -1000 & 3500 \\ -1000 & 62500 & -500 \\ 3500 & -500 & 32000 \end{bmatrix} \text{ kg \cdot m}^2 \tag{4}
\]

(b) Use the result of (a) to find the angular velocity in the principal frame as \(\omega_F\) if the angular velocity in the current \(B\) body frame is \(\omega_B = [0.15, -0.25, -0.8]^T\) rad/s.
Figure 1: Rigid body with a body-fixed reference frame $B$ with its origin at the center of mass

(c) Show that

$$\text{Tr } B\{I_c\} = \text{Tr } F\{I_c\},$$

where $\text{Tr}(\cdot)$ denotes the trace of a matrix.

(d) Find a principal rotation angle $\Phi$ and axis $\hat{e}$ for the direction cosine matrix $[FB]$.

Problem 4.

Consider a rigid body as depicted in Fig 1. Let $B\{I_c\}$ be the inertia matrix of the body frame related to its center of mass as

$$[I_c] = \int (\vec{p}^T \vec{p} I_3 - \vec{p} \vec{p}^T) dm,$$

where $\vec{p}$ denotes the position vector of $dm$ relative to the center of mass. Also let $[I_o]$ be the inertia matrix of the body relative to an arbitrary point $O$ as

$$[I_o] = \int (\vec{r}^T \vec{r} I_3 - \vec{r} \vec{r}^T) dm,$$

where $\vec{r}$ denotes the position vector of $dm$ relative to point $O$. The position vector of the center of mass from $O$ is denoted by $\vec{d}$ and $\vec{r} = \vec{d} + \vec{p}$. Then, verify the parallel axis theorem

$$[I_o] = [I_c] + m \left[ \vec{d}^T \vec{d} I_3 - \vec{d} \vec{d}^T \right],$$

where $m$ denotes the mass of the body.

Problem 5.

The principal inertia of a rigid satellite are given by

$$[I_c] = \text{diag}(210, 200, 118) \text{kg} \cdot m^2$$

At time $t_0$, the body angular velocity is $(0.2, 0.15, -0.18)^T$ rad/s. Numerically solve the resulting torque-free motion for 30 seconds and plot the resulting attitude in terms of the (3-2-1) Euler angles. Use the initial Euler angles to
be zeros. The kinematics equation for the (3-2-1) Euler angles is given by

\[
\begin{bmatrix}
\dot{\psi} \\
\dot{\theta} \\
\dot{\phi}
\end{bmatrix} = \frac{1}{\cos \theta} \begin{bmatrix}
0 & \sin \phi & \cos \phi \\
0 & \cos \phi \cos \theta & -\sin \phi \cos \theta \\
\cos \theta & \sin \phi \sin \theta & \cos \phi \sin \theta
\end{bmatrix} \begin{bmatrix}
\omega_1 \\
\omega_2 \\
\omega_3
\end{bmatrix}.
\]

(10)

**Problem 6.** Consider an inertia matrix of the form

\[
\begin{bmatrix}
I_{11} & I_{12} & I_{13} \\
I_{21} & I_{22} & I_{23} \\
I_{31} & I_{32} & I_{33}
\end{bmatrix},
\]

\[
\begin{bmatrix}
I_{11}' \quad 0 \quad 0 \\
0 \quad I_{22}' \quad 0 \\
0 \quad 0 \quad I_{33}'
\end{bmatrix}
\]

(11)

which is symmetric and positive definite. (All of the eigenvalues of a symmetric matrix are real. A symmetric matrix is said to be positive definite if all its eigenvalues or all its leading principal minors are positive.)

(a) Show that the moments of inertia are interrelated by

\[
I_{11} + I_{22} > I_{33}, \quad I_{22} + I_{33} > I_{11}, \quad I_{33} + I_{11} > I_{22}
\]

(12)

which are called the triangle inequalities.

(b) Show that

\[
I_{11}I_{22} + I_{22}I_{33} + I_{33}I_{11} - I_{12}^2 - I_{23}^2 - I_{31}^2 = I_{11}'I_{22}' + I_{22}'I_{33}' + I_{33}'I_{11}'
\]

(13)

\[
I_{11}I_{22}I_{33} + 2I_{12}I_{23}I_{31} - I_{11}I_{23}^2 - I_{22}I_{31}^2 - I_{33}I_{12}^2 = I_{11}'I_{22}'I_{33}'
\]

(14)