Another very useful concept is that of "convergence."

**Def:** An infinite sequence of vectors in a normed space \( V \), \( \{v_n\}_{n=1}^{\infty} \), is said to converge to a vector \( v^* \) if \( \|v_n - v^*\| \to 0 \) as \( n \to \infty \).

In this case we write \( v_n \to v^* \).

Note that \( v^* \) is unique!

If \( v_n \to v^* \) and \( v_n \to w^* \),

Then \( \|v^* - w^*\| = \|v_n - v^* + v^* - w^*\| \leq \|v_n - v^*\| + \|v^* - w^*\| \to 0 \)

\( \Rightarrow \|v^* - w^*\| = 0 \Rightarrow v^* = w^* \)
Prop: A set is closed if every convergent sequence with element in the set has a limit point in the set.

(see Humberger p. 27)

In practice it is difficult to determine if every convergent sequence has a limit point in the set.

For this an additional concept called

Completeness

often becomes handy. Consider a convergent sequence

\[ v_n \to v^* \]

\[ v_1, v_2, v_3, \ldots, v_n \to v^* \]

It is clear that since \( v_n \to v^* \), \( v_n \) and \( v_m \) as \( n, m \to \infty \) should get closer too!
Def: A sequence is called a Cauchy sequence if \( \epsilon > 0, \exists N \) s.t. \( \forall n, m > N \)

\[ ||v_n - v_m|| < \epsilon \]

or that \( ||v_n - v_m|| \to 0 \) as \( n, m \to \infty \).

In practice it is easier to establish that a sequence is Cauchy \( \Rightarrow \) since we know \( v_n, v_m \)!

A normed vector space is called complete if every Cauchy sequence is convergent (to a limit in the vector space).

A complete normed space is called a Banach space.
An inner product space that is complete with respect to the norm induced by the inner product is called a Hilbert space.
So last time we started doing some analysis.

Analytic concepts: convergence, limits, continuity, ...

Topological concepts: open sets, closed sets, interior, closure, ...

(point set)

We talked about convergent sequences & Cauchy sequences

\[ v_1, \ldots, v_n, \ldots \to v^* \]

\[ \| v_n - v^* \| \to 0 \quad n \to \infty \]

If a set is closed, (it is equal to its closure),

& \( v_n \) is a sequence in the set \( \& v_n \to v^* \), then \( v^* \)

is in the set. Why? \( v^* \) should be in the closure of the set.

\( \forall \epsilon > 0, \quad B(v^*, \epsilon) \) has an intersection with the set \( (v_n's \) are in the set. \)
This is an important property of closed sets:

\{ \text{All convergent sequences converge to} \}
\{ \text{a vector in the set} \}

Another concept was the notion of completeness:

In a complete vector space every Cauchy sequence is convergent to a vector in the vector space.

**Def**: A complete normed vector space is called a Banach space.
Example: Consider $V$ as the vector space of sequences

$$V = \{ x_1, x_2, \ldots, x_n, 0, \ldots \}$$

with finitely many zeros.

Let $\|v\| = \max_i |x_i|$

Can we construct a Cauchy sequence in $V$ that is not convergent?

Yes: let

$$v_k = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{k-1}, 0, 0, 0, \ldots\} \quad k \geq 1$$

i.e.,

$$v_1 = \{1, 0, 0, 0, \ldots\}$$

$$v_2 = \{1, \frac{1}{2}, 0, 0, \ldots\}$$

$$v_3 = \{1, \frac{1}{2}, \frac{1}{3}, 0, 0, \ldots\}$$
Does \( v_k \to v^* \) in \( V \)? No!

Is \( v_k \) a Cauchy sequence? Yes!

\[
\| v_n - v_m \| = \max \{ \| v_n \|, \| v_m \| \} \to 0 \quad \text{as} \quad n, m \to \infty.
\]

Some examples of \( C \) Banach spaces:

\( C[0,1] \): continuous functions on the interval \([0,1]\), with \( \| u \| = \max_{t \in [0,1]} |u(t)| \).

Note that I am using "max" than "sup".

A continuous function achieves its max \& min on a closed interval.
Completeness depends critically on the norm:
if $C[0,1]$ is given the norm
$$\|v(t)\| = \int_0^1 |v(t)| \, dt$$
then it is can be shown
that it is not complete. (Luenberger p. 34).

The notion of closedness & completeness are
closely related; for Banach Spaces:

Thm: In a Banach Space, a subset is
closed iff it is complete.
Proof: If a subset is complete, every Cauchy sequence is convergent to a vector in the subset. Since every convergent sequence is Cauchy, the set is closed.

On the other hand, if the set is closed, every Cauchy sequence in the Banach space has to be convergent. If the set is closed, the limit belongs to the set.
An important fact is that in any normed space, any finite dimensional subspace is complete.

(see p. 38 of Eveshigler)

Let me show you how this works out nicely in a Hilbert space (a complete inner product space).

The inner product allows us to project:

\[ \lim_{n \to \infty} v_n \Rightarrow v^* \text{ then } \|v_n - v^*\| \to 0. \]

However, \[ \|v^* - \Pi_{\text{subspace}}(v^*)\| \leq \|v_n - v^*\| \quad \forall n. \]

\[ \Rightarrow v^* = \Pi_{\text{subspace}}(v^*) \Rightarrow v^* \in \text{subspace}. \]
In the Hilbert space, all closed subspaces are complete.

\{ A finite dimensional subspace is a Hilbert space if closed (and complete) \}
We will start to discuss more systematically vector spaces whose elements are functions and sequences in these vector spaces are not finite dimensional.

i) Infinite sequences:

\[ x = \xi_1, \xi_2, \ldots, \xi_n, \ldots \]

\[ \ell_p \text{ space: } \sum_{i=1}^{\infty} |\xi_i|^p < \infty. \]

If \( p = 2 \): square summable sequences:

\[ \sum_{i=1}^{\infty} |\xi_i|^2 < \infty. \]

\[ \ell_{\infty} : \sup_{i} |\xi_i| < \infty. \]
In this case we can define the norm in the $l_p$ space as:

$$
\|x\|_p = \left( \sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}
$$

And for $l_\infty$ as:

$$
\|x\|_\infty = \sup_i |x_i|.
$$

ii) The "function" version of $l_p$ is often referred to as $L_p$: the vector space of real-valued functions on $[a, b]$ for which

$$
\int_{a}^{b} |f(t)|^p \, dt < \infty
$$

This is called $L_p[a, b]$. 
In this case we can & make $L^p$ is normed space:

$$\|f\|_p = \left( \int_a^b |f(t)|^p \, dt \right)^{1/p}$$

the $L^p[a,b]$ is the vector space of real-valued functions on $[a,b]$ s.t.

$$\sup_{t \in [a,b]} |f(t)| < \infty$$

so for example $f(t) = t^2 \in L^\infty[0,1]$ but $\notin L^\infty[0,\infty)$
of particular interests are $\ell_2$ and $\ell_2$. Why?

They are Hilbert spaces!

$L_2[a,b]$: $\|f\|_2 = \left(\int_a^b |f(t)|^2 \, dt\right)^{1/2}$

$\langle f, g \rangle = \int_a^b f(t)g(t) \, dt$.

$\ell_2 E$: $\|x\|_1 = \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{1/2}$

$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i$

$\{x_1, x_2, \ldots, x_n, \ldots\}$

$\{y_1, y_2, \ldots, y_n, \ldots\}$
We will look at some of the applications of the Hilbert space of $L^2$ in control theory.

Before we do that let us look at the projection theorem in the context of Hilbert spaces:

If $M$ is a closed subspace in a Hilbert space, $H$, if $x \in H$, then a unique $m_0 \in M$, s.t.

$\|x - m_0\| \leq \|x - m\| \quad \forall \quad m \in M$.

The unique $m_0$ is characterized by:

$x - m_0 \bot M$. 

projection of $x$ onto $M$. 

Proof: Let us first show that \( x - m_0 \perp M \).

Suppose not, that is, there exists \( m \in M \) such that \( m \not= x - m_0 \).

Normalize \( m \) so that \( \| m \| = 1 \). Thus

\[
\langle x - m_0, m \rangle = 0
\]

Consider \( m_0 + \delta m \). Then:

\[
\| x - m \| ^2 = \| x - m_0 - \delta m \| ^2 = \| x - m_0 \| ^2 - \langle x - m_0, \delta m \rangle - \langle \delta m, x - m_0 \rangle + \| \delta m \| ^2
\]

Therefore

\[
\| x - m \| ^2 = \| x - m_0 \| ^2 - \| \delta m \| ^2 < \| x - m_0 \| ^2
\]

Thus \( x - m_0 \perp M \).
let us now show uniqueness of this $m_0$.

Consider $m \in \mathcal{M}$. Then

$$
\|x - m\|^2 = \|x - m_0 + m_0 - m\|^2
$$

$$
= \|x - m_0\|^2 + \|m_0 - m\|^2
$$

so if $m_0 \neq m$, $\|x - m\|^2 > \|x - m_0\|^2$, that is $m_0$ is unique.

Now how does closedness of the subspace come into picture?

In the above argument we said that if $m_0$ exists, then $x - m_0 \in \mathcal{M}$.

But who said $x - m_0$ is a minimizing vector?