Recall again our "class picture"

Two additional structure on this setup:

Norm:
\[ \| \cdot \| : V \to \mathbb{R}_+ \]
\[ \|x\| = 0 \iff x = 0 \]
\[ \|\alpha x\| = |\alpha| \|x\| \]
\[ \|x + y\| \leq \|x\| + \|y\| \]

Metric:
\[ g : V \times V \to \mathbb{R}_+ \]
\[ g(x, y) = 0 \iff x = y \]
\[ g(x, y) = g(x, y) \]
\[ g(x, z) \leq g(x, y) + g(y, z) \]
a norm \( \| \cdot \| \) induces a metric 
but a metric doesn't have to be induced by a norm.

\[ ex: \]

1) \( d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases} \)

2) \( d(x, y) = \max_{i \in \mathbb{R}^n} |x_i - y_i| \)

3) \( d(f, g) = \sup_{a \leq t \leq b} \{ |f(t) - g(t)| \} \) 

\( f, g \) cont. functions 
on \([a, b]\)

1) is not induced by a norm \( \rightarrow \) homogeneity is the problem

2, 3 are induced by a norm but not an inner product
Can we define a norm on $L(V)$ or $L(W, V)$? Yes! One way is through an induced operator norm:

$$
\|T\| = \sup_{\|v\|=1} \|Tv\|_W
$$

$$
= \sup_{v \neq 0} \frac{\|Tv\|_W}{\|v\|_V}
$$

an amplification induced by $T$.

Let us see what is this operator norm for $A \in \mathbb{R}^n$

$$
\|A\|_2 = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}
$$

$$
\rightarrow \|A\|^2 = \sup_{x \neq 0} \frac{x^T A^T A x}{x^T x} = \sigma_{\max}(A)^2
$$

$$
\rightarrow \|A\| = \sigma_{\max}(A)
$$
let us quickly check one of the properties of the norm for the induced operator norm:

\[ \| T + S \| \leq \| T \| + \| S \| \]

Consider

\[ \| (T + S)v \| = \| Tu + Sv \| \leq \| Tu \| + \| Sv \| \leq \| T \| \| u \| + \| S \| \| v \| \leq (\| T \| + \| S \| ) \| v \|. \]

Therefore

\[ \| T + S \| = \sup_{\| v \| = 1} \| (T + S)v \| \leq \| T \| + \| S \|. \]

*
In fact an induced operator norm has an additional property: given \( S, T \in \mathcal{L}(V) \)

\[ \forall v \in V : \quad \| ST v \| \leq \| S \| \| T v \| \leq \| S \| \| T \| \| v \|. \]

\& therefore

\[ \| ST \| \leq \| S \| \| T \|. \]

This is called the multiplicative property of induced operator norm.
So our picture is now:

What does this additional structure do for us? Well, the first concept related to this structure is that of open/closed sets:

A set \( \mathcal{P} \) is a normal vector space if

\[ \forall \vec{v}, \vec{w} \in \mathcal{H} \text{ s.t. } \| \vec{v} - \vec{w} \| < \varepsilon \text{ implies that } \vec{v} \in \mathcal{P} \]
We can also define an open set through the notion of the interior of the set: \( \mathbf{v} \in H \) is in the interior of \( P \) if \( \mathbf{v} \in P \), \( \exists \varepsilon > 0 \) s.t. all vectors \( \mathbf{w} \) for which \( \| \mathbf{v} - \mathbf{w} \| < \varepsilon \) are in \( P \).

The set \( \{ \mathbf{w} \mid \| \mathbf{v} - \mathbf{w} \| < \varepsilon \} \) is called the \( B(\mathbf{v}, \varepsilon) \) open ball centered at \( \mathbf{v} \) w/ radius \( \varepsilon \).

The interior of a set \( P \subseteq H \) is denoted by \( \overline{P} \).

The other concept is that of a closed set.

Def: A point vector \( \mathbf{v} \in H \) is called the closure point of \( P \) if \( \forall \varepsilon > 0, \exists \mathbf{w} \in P \) s.t. \( \| \mathbf{v} - \mathbf{w} \| < \varepsilon \) and \( B(0, \varepsilon) \) open \( B(0, \varepsilon) \) for each \( \varepsilon > 0 \).

If \( P = \overline{P} \), \( P \) is called closed. The set of all closure points of \( P \) is called the closure of \( P \), \( \overline{P} \).
Prop: The complement of an open set is closed.

Let \( P \) be an open set & its complement
\[ \tilde{P} = \{ x \mid x \notin P \}. \]

Can \( \tilde{P} \) contain all its closure points?
In other words, is it possible that \( P \) contains a closure point of \( \tilde{P} \)?

No: since \( P \) is open, \( \forall x \in P, \exists \varepsilon > 0 \) s.t. \( B(x, \varepsilon) \subseteq P \).

So no point of \( P \) can be the closure point of \( \tilde{P} \).

\[ \Rightarrow \tilde{P} \text{ contains all its closure points.} \]
\[ \Rightarrow \tilde{P} \text{ is closed.} \]
Some observations:

\[
\begin{align*}
\{ \text{open sets} \} & \cap \{ \text{finite closed sets} \} = \{ \text{open closed} \} \\
\{ \text{open sets} \} \cup \{ \text{closed sets} \} & = \{ \text{open closed} \} \\
\text{finite}
\end{align*}
\]

One can also define the notion of relative interior w.r.t. a subspace or an affine space:

- open set
- closure of \( B \)
- open set

\( M \) is not open w.r.t. of \( \mathbb{R}^2 \)

open w.r.t. \( M \) (relative) to
Another very useful concept is that of "convergence."

**Def:** An infinite sequence of vectors in a normed space \( V \), i.e. \( \{v_n\}_{n=1}^{\infty} \), is said to converge to a vector \( v^* \) if \( \|v_n - v^*\| \to 0 \) as \( n \to \infty \).

In this case we write,

\[ v_n \to v^* \]

a limit point of the sequence.

Note that \( v^* \) is unique!

if \( v_n \to v^* \)

\[ v_n \to w^* \]

Then \( \|v^* - w^*\| = \|v_n - v^* + v^* - w^*\| \leq \|v_n - v^*\| + \|v^* - w^*\| \to 0 \)

\[ \|v^* - w^*\| = 0 \to v^* = w^* \]