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On Strong Structural Controllability of Networked Systems: A Constrained Matching Approach

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Abstract—**This paper examines strong structural controllability of linear-time-invariant networked systems. We provide necessary and sufficient conditions for strong structural controllability involving constrained matchings over the bipartite graph representation of the network. An $\mathcal{O}(n^2)$ algorithm to validate if a set of inputs leads to a strongly structurally controllable network and to find such an input set is proposed. The problem of finding such a set with minimal cardinality is shown to be NP-complete. Minimal cardinality results for strong and weak structural controllability are compared.**

Index Terms—**Strong structural controllability, Network controllability, Network observability, constrained matchings**

I. Introduction

Complex dynamic networks are an integral part of our natural world, such as biological, chemical and social networks, and our technological world with designed networks, such as the internet, power grids, and robotic networks. In recent years, there has been a myriad of research in the area of network systems [1], [2], [3]. Of increasing importance is the manipulation and monitoring of such networks. The cornerstone of effective control and observation of networked systems is the appreciation of the interplay between system performance and network structure.

Research in weak and strong structural controllability (s-controllability) has had a rich history since its inception [4], [5], and by its very nature exposes the role of structure in network control. Structural controllability establishes generic (weak) and complete (strong) controllability of a network based solely on the direct coupling between nodes appearing as a distinct pattern of zeros in the network dynamics. This is irrespective of the magnitude of these couplings.

The attraction of s-controllability is that, independent of variations in the coupling strength, controllability can be guaranteed. This provides network controllability robust to parameter uncertainty and floating point errors. Further, unlike controllability, methods to establish s-controllability are numerically stable. Weak and strong s-controllability also provide lower and upper bounds, respectively, on the cardinality of a minimum input set for controllability.

Although it is atypical that a weakly s-controllable network has a coupling strength realization rendering it uncontrollable, there are systems that are unconsciously designed with such coupling. An example with such a coupling is unweighted undirected graphs such as the aforementioned consensus-based networks. Such homogeneity generates symmetry in the network which typically renders such a system uncontrollable [6]. For these cases, strong s-controllability presents itself as a useful alternative to weak s-controllability.

Recently, a result by Liu *et al.* has appeared linking weak s-controllability to matchings in a bipartite graph representation of the network [7]. This provides an attractive and efficient way to form generically controllable input sets. The question addressed in the paper is whether a similar matching-based method can be found to efficiently check and find completely controllable input sets. Steps in this direction were undertaken by Olesky *et al.* for the single entry case linking constrained t-matchings to strong s-controllability [8]. Reinschke *et al.* established an $\mathcal{O}(n^3)$ method to check an input set is strongly s-controllable [9]. Moreover, Reinschke *et al.* conjectured a graph-theoretic method involving spanning cycles to establish strong s-controllability which was recently refined and proven by Jarczyk *et al.* [10].

In this paper, we provide necessary and sufficient constrained matching conditions for strong s-controllability. This result has implications for both checking and finding strongly s-controllable input sets. In this direction, the matching conditions are applied to check an input set is strongly s-controllable in $\mathcal{O}(n^2)$. It is established that finding a minimum cardinality input set is NP-complete. A greedy $\mathcal{O}(n^2)$ algorithm is supplied to provide a strongly s-controllable input set which has been shown through simulation to perform well.

The organization of the paper is as follows. We commence with the introduction of notation, graphs, bipartite matching and pattern matrices. We present the model that we are examining and some existing results in strong s-controllability. We provide constrained matching conditions equivalent to strong s-controllability. The problem of finding a strongly s-controllable input set is investigated. Finally, we propose an algorithm to validate strong s-controllability and find a strongly s-controllable input set.

II. Notations and Preliminaries

We provide a brief background on constructs that will be used in this paper. For column vector $v \in \mathbb{R}^p$, v_i or $[v]_i$ denotes the ith element, e_i denotes the column vector which contains all zero entries except $[e_i]_i = 1$. For matrix $M \in \mathbb{R}^{p\times q}$, $[M]_{ij}$ denotes the element in its ith row and jth column. We form the submatrix $A(\alpha|\cdot)$ from $A \in \mathbb{R}^{m\times n}$, where $\alpha \subseteq \{1, \ldots, m\}$, by removing rows in α.

Graphs: A succinct way to represent the interactions of agents in a network is through a graph. A graph $\mathcal{G} = (V, E)$

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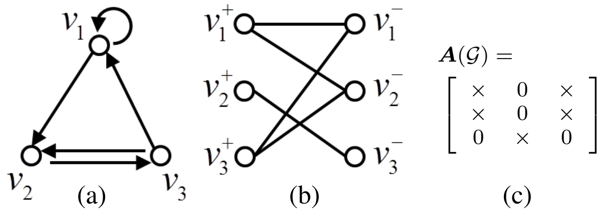

Figure 1. (a) Graph $\mathcal{G}$ (b) Equivalent bipartite graph and (c) Pattern matrix.

is defined by a node set V with cardinality n, the number of nodes in the graph which represent the agents in the network, and an edge set E comprising of pairs of nodes which represent the agents interactions, i.e., agent i affects agent j's dynamics if there an edge from i to j, i.e., $(i, j) \in E$. A self-loop is an edge $(i, i) \in E$ and the set of nodes with self-loops is V_s. The adjacency matrix $A(\mathcal{G})$ is a matrix representation of $\mathcal{G}$ with $[A(\mathcal{G})]_{ji} = 1$ for $(i, j) \in E$ and $[A(\mathcal{G})]_{ji} = 0$, otherwise. If every node in V can (not) be reached along a directed path from at least one node in V_1, then V_1 is input *(in)accessible.*

A bipartite graph $\mathcal{H} = (V^+, V^-, E)$ is an undirected graph[1] on independent node sets V^+ and V^- where the edge set E connects nodes in V^+ to V^-. A bipartite graph representation of a matrix $M \in \mathbb{R}^{p\times q}$ can be formed by setting $V^+ = \{1, \dots, q\}$ and $V^- = \{1, \dots, p\}$ and setting an edge $\{i, j\} \in E$ if and only if $[M]_{ij} \neq 0$. The bipartite graph $\mathcal{H} = (V^+, V^-, E)$ associated with $\mathcal{G}$ can be formed in this way over $A(\mathcal{G})$. Figure 1 demonstrates an equivalent bipartite graph $\mathcal{H}$ of a graph $\mathcal{G}$.

Bipartite Matching and Pattern Matrices: A set of t edges $\left(i_1^+, i_1^-\right), \dots, \left(i_t^+, i_t^-\right)$ in bipartite graph $\mathcal{H}$ is said to be a *t-matching* if $I^+ = \left\{i_1^+, \dots, i_t^+\right\} \subseteq V^+$ and $I^- = \left\{i_1^-, \dots, i_t^-\right\} \subseteq V^-$. Such an t-matching is said to be a *constrained t-matching* (or uniquely restricted matching) if it is the only t-matching between I^+ and I^-. Those nodes in $\mathcal{H}$ that are in I^+ or I^- are referred to as *matched*, and those not are *unmatched*. A matching is *T-less* if it contains no edges of the form (j^+, j^-), where j in in the set T. A T-less matching where $T = \{1, \dots, \min\{|V^+|, |V^-|\}\}$, is referred to as a *self-less* matching. A (constrained) (T-less) t-matching in $\mathcal{H}$ is *maximum* if there is no (constrained) (T-less) s-matching in $\mathcal{H}$ with $s > t$.

A *pattern matrix* $\boldsymbol{A}$ (or Boolean structure matrix) is a matrix with each element either a zero or a star, denoted throughout by an $\times$. A numerical matrix A is called a *realization* of the pattern $\boldsymbol{A}$ if A can be obtained by assigning nonzero numerical values to the star entries of $\boldsymbol{A}$, in short $A \in \boldsymbol{A}$. A modification of a pattern matrix can be made by placing stars along the diagonal of $\boldsymbol{A}$ and is denoted by $\boldsymbol{A}_\times$. The pattern matrix $\boldsymbol{A}(\mathcal{G})$ of a graph $\mathcal{G}$ can be formed by replacing the nonzero entries of $A(\mathcal{G})$ with stars (see Figure 1 for example). We say that a matrix A or pattern matrix $\boldsymbol{A}$ has a (constrained) t-matching if the associated bipartite graph $\mathcal{H}$ has a (constrained) t-matching.

[1]A graph is undirected if $(i, j) \in E$ implies $(j, i) \in E$. Special undirected graphs are the complete graph which has an edge between every node and the path graph where $(i, j) \in E$ if and only if $|i - j| = 1$.

III. Model

Commonly, for a linear-time-invariant networked system of single-integrator agents, the graph $\mathcal{G}$ characterizes the zero structure of the state matrix through $A(\mathcal{G})$, and input and output points to the network appear in the input and output matrices [11]. In other terms, the graph $\mathcal{G}$ and input node set S define pattern matrices and some realization of these pattern matrices define the dynamics. Specifically, the state matrices are realizations of the $n \times n$ pattern matrix $\boldsymbol{A}(\mathcal{G})$. The input node set $S = \{i_1, i_2, \dots, i_m\}$, where $m \leq n$, define the $n \times m$ pattern matrix $\boldsymbol{B}(S)$ formed by setting the nonzero entries of $[e_{i_1}, e_{i_2}, \dots, e_{i_m}] \in \mathbb{R}^{n\times m}$ to stars. Similarly, for S the output node set, the output matrix forms the pattern matrix $\boldsymbol{B}(S) = \boldsymbol{C}(S)^T$.

In this paper, we explore structural controllability (and observability) for a system of the form

$$\dot{x}(t) \quad = \quad Ax(t) + Bu(t), \quad y(t) = Cx(t)$$

where $A \in \boldsymbol{A}(\mathcal{G})$, $B \in \boldsymbol{B}(S_1)$ and $C \in \boldsymbol{C}(S_2)$. For succinctness, if $\mathcal{G}$ and S are clear from the context then $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ will replace $\boldsymbol{A}(\mathcal{G})$, $\boldsymbol{B}(S_1)$ and $\boldsymbol{C}(S_2)$, respectively. Henceforth, we shall focus on structural controllability of the pair $(\boldsymbol{A}, \boldsymbol{B})$, noting that results can be similarly applied to observability of the pair $(\boldsymbol{A}, \boldsymbol{B}^T)$ through duality.

IV. Structural Controllability

Lin [4] defined a pair $(\boldsymbol{A}, \boldsymbol{B})$ as weakly structurally controllable (s-controllable) if it admits a some controllable numerical realization (A, B). Mayeda and Yamada [5] adapted this, defining the pair $(\boldsymbol{A}, \boldsymbol{B})$ as strongly s-controllable system if *all* numerical realizations (A, B) are controllable. We state that for a given $\boldsymbol{A}$, the input set S is weakly (strongly) s-controllable if the pair $(\boldsymbol{A}, \boldsymbol{B}(S))$ is weakly (strongly) s-controllable.

Recently, Liu *et al.* [7] linked matchings in the bipartite graph $\mathcal{H}$ corresponding to $\mathcal{G}$ to the weak s-controllability of $(\boldsymbol{A}(\mathcal{G}), \boldsymbol{B}(S))$. The following summarizes this result.

Theorem 1. *The pair $(\boldsymbol{A}, \boldsymbol{B}(S))$ is weakly s-controllable from a nonempty m-input set S if and only if $\boldsymbol{A}(S|\cdot)$ has a $(n-m)$-matching and S is input accessible.*

The strength of this result is that the structure of the graph can be linked to a controllable input set. Further, this graph feature can be efficiently computed via maximum matching algorithms for bipartite graphs in $\mathcal{O}\left(\sqrt{n}\,|E|\right)$ time [12], coupled with a depth first search to determine input accessibility in $\mathcal{O}\left(|E|\right)$ time.

Noting the similarities in the definitions for weak and strong s-controllability, it is not surprising that structural features that are equivalent to weak s-controllability share similarities to structural features that are equivalent to strong s-controllability. It is with this in mind that we explore the role of t-matchings in strong s-controllability. To this end, Reinschke *et al.* [9] conjectured an algebraic result, later proved by Jarczyk *et al.* [10], for strong controllability. In Section V, we will show that Theorem 3 has an equivalent

t-matching realization. The following construct is necessary for the result.

Definition 2. A structured pair $(\boldsymbol{A}, \boldsymbol{B})$ is said to be in Form III if there exists two permutation matrices P_1 and P_2 such that

$$P_1\left[\boldsymbol{A}, \boldsymbol{B}\right] P_2 = \begin{bmatrix} \otimes & \cdots & \otimes & \times & 0 & \cdots & 0 \\ \vdots & & \vdots & \ddots & \times & \ddots & \vdots \\ & & & & \ddots & \ddots & 0 \\ \otimes & \cdots & \otimes & \cdots & \cdots & \otimes & \times \end{bmatrix},$$

where the $\times$-elements denote the location of star elements and the $\otimes$-elements denotes the location of either zero or star elements.

The aforementioned result follows.

Theorem 3. *[9] The structured pair $(\boldsymbol{A}, \boldsymbol{B})$ is strongly s-controllable if and only if*

1. the pair $(\boldsymbol{A}, \boldsymbol{B})$ is of Form III,

2. and the matrix $(\boldsymbol{A}_\times, \boldsymbol{B})$ can be transformed into Form III in such a way that the $\times$-elements do not correspond to the diagonal elements in $\boldsymbol{A}$ that were stars.

V. TESTING INPUTS FOR STRONG S-CONTROLLABILITY

The connection of Theorem 3 is through the following constrained matching property.

Theorem 4. *[13] Let $\mathcal{H} = (V^+, V^-, E)$ be a bipartite graph. A t-matching M is constrained if and only if we can order the nodes of $V^+ = \{v_1^+, \ldots, v_n^+\}$, and $V^- = \{v_1^-, \ldots, v_m^-\}$, such that $\left(v_i^+, v_i^-\right) \in M$, $1 \leq i \leq t$, and $\left(v_i^+, v_j^-\right) \notin E$ for $1 \leq j < i \leq t$.*

The following theorem presents the promised equivalent constrained matching result to Theorem 3.

Theorem 5. *Let S be an input set with cardinality $m \leq n$. The pair $(\boldsymbol{A}, \boldsymbol{B}(S))$ is strongly s-controllable if and only if $\boldsymbol{A}(S|\cdot)$ has a constrained $(n-m)$-matching and $\boldsymbol{A}_\times(S|\cdot)$ has a constrained V_s-less $(n-m)$-matching.*

Proof: From Theorem 4, a pair $(\boldsymbol{A}, \boldsymbol{B}(S))$ is of Form III (the first condition in Theorem 3) if and only if $\boldsymbol{A}(S|\cdot)$ has a constrained $(n-m)$-matching. The second condition in Theorem 3 is equivalent to the existence of a constrained V_s-less $(n-m)$-matching in $\boldsymbol{A}_\times(S|\cdot)$. This is apparent by isolating the associated constrained matchings codified in the nonzero diagonal of $P_1\left[\boldsymbol{A}, \boldsymbol{B}\right] P_2$. The elements on this diagonal will be of the form $\left\{a_{i_1^+ j_1^-}, \ldots a_{i_{n-m}^+} a_{j_{n-m}^-}, b_{1k_1}, \ldots, b_{mk_m}\right\}$, where $S = \{k_1, \ldots, k_m\}$ and correspond to a constrained $(n-m)$-matching $\left\{i_1^+, j_1^-\right\}, \ldots, \left\{i_{n-m}^+, j_{n-m}^-\right\}$. Similarly, for $P_1\left[\boldsymbol{A}_\times, \boldsymbol{B}\right] P_2$ the elements corresponding to V_s appear below the diagonal and so do not appear in the constrained $(n-m)$ matching, making it V_s-less. ∎

Consequently, a check that an input set S is strongly s-controllable reduces to the problem of finding constrained $(n-m)$-matchings. This is demonstrated in the following example.

Example 6. Returning to Figure 1, for $i = 1$ and 2 (but not 3), $\boldsymbol{A}(S|\cdot)$ with input set $S = \{i\}$ has a constrained 2-matching, satisfying the first condition in Theorem 5. For set $V_s = \{1\}$, the pattern matrix $\boldsymbol{A}_\times$ has constrained V_s-less 2-matchings, $\left\{\left(v_1^+, v_2^-\right), \left(v_2^+, v_3^-\right)\right\}$ and $\left\{\left(v_2^+, v_3^-\right), \left(v_3^+, v_1^-\right)\right\}$. Thus $\boldsymbol{A}_\times(S|\cdot)$ has a constrained V_s-less 2-matchings satisfying the second condition in Theorem 5. Therefore, the pair $(\boldsymbol{A}, \boldsymbol{B}(\{1\}))$ and $(\boldsymbol{A}, \boldsymbol{B}(\{2\}))$ are strongly s-controllable.

The conditions in Theorem 5 require a validation that there exists a constrained n-matching. A maximum bipartite matching can be found deterministically in $\mathcal{O}(\sqrt{n}\,|E|)$ [12], and testing whether a given bipartite matching is constrained can be checked in $\mathcal{O}(n + |E|)$ [13].

Reinschke *et al.* [9] provided a $\mathcal{O}(n^3)$ algorithm to check the conditions in Theorem 3. We have used a similar approach in Algorithm 1 but performed the check in $\mathcal{O}(n^2)$, reducing computation by tracking the column sums through the vectors σ and $\sigma_\otimes$. Due to its similarity to [9] we present the algorithm here without proof. This algorithm is further developed in the following section, and so if only validation of strong s-controllability is required then the process can be terminated at the lines marked with asterisks, in which case the set S is not strongly s-controllable. For the case where S is strongly s-controllable, this will be validated once the algorithm has run its course.

A. Minimum Cardinality Input Sets

In this section we explore the smallest strongly s-controllable input set for a special class of networks. First, a general result.

Proposition 7. *There exists no graph that is strongly s-controllable from all single inputs sets $S = \{i\}$, for $i = 1, \ldots, n$.*

Proof: Assume otherwise, by Theorem 5, $\boldsymbol{A}_\times(\{i\}|\cdot)$ has a constrained V_s-less $(n-1)$-matching for every $i \in \{1, \ldots, n\}$. Using the property that a $m \times n$ pattern matrix $\boldsymbol{M}$ is full rank if and only if it contains a constrained $\min(m, n)$-matching ([14] Theorem 3.9), then every realization $A \in \boldsymbol{A}_\times$ has a rank $n-1$ submatrix when an arbitrary row is removed. Therefore, every such A has rank n and so $\boldsymbol{A}_\times$ has a constrained n-matching from the same property. By Theorem 4, there exists permutation matrices P_1 and P_2 such that $P_1 \boldsymbol{A}_\times P_2$ is lower triangular with nonzero diagonal, i.e., the elements corresponding to V_s must lie below the diagonal. As there must be exactly one term of the form a_{ii} in every row and column of $P_1 \boldsymbol{A}_\times P_2$, then the diagonal of $P_1 \boldsymbol{A}_\times P_2$ are the elements $a_{11}, \ldots, a_{nn}$ of $\boldsymbol{A}_\times$. Therefore V_s is empty otherwise one of its elements would lie on the diagonal. Similar to the argument for $\boldsymbol{A}_\times$, the pattern matrix $\boldsymbol{A}$ must also be triangulizable with nonzero diagonal elements corresponding to $a_{11}, \ldots, a_{nn}$, but if V_s is empty, then $a_{ii} = 0$ for all i and contradiction the fact that $\boldsymbol{A}$ has nonzero diagonal elements. ∎

Proposition 7 is in stark contrast to the weakly s-controllable where there are many networks which exhibit single input controllability from an arbitrary node. One such class of networks that falls in this category for weak s-controllability is the family of self-damped connected networks. These are networks where every node's state directly damps itself, i.e., for every node i, $\dot{x}_i = \alpha_i x_i + \sum_{j \neq i} \alpha_j x_j$, where $\alpha_i \neq 0$. The popular consensus (Laplacian) dynamics fall into this class for connected undirected graphs.

The following two propositions pertain to self-damped undirected networks. Proposition 8 illustrates the rarity of single strongly s-controllable inputs.

Proposition 8. *The only connected self-damped undirected network strongly s-controllable from a single input is the path graph, controllable from either end node.*

Proof: If $\boldsymbol{A}$ is self-damped and $S = \{i\}$ then $(\boldsymbol{A}, \boldsymbol{B}(S))$ is strongly s-controllable if and only if there exists a permutation matrix P such that $P\boldsymbol{A}P^T$ is unreduced upper-Hessenberg[2] and $P\boldsymbol{B}(S) = \boldsymbol{B}(S)$ ([8] Theorem 2.4). The first condition is equivalent to the graph bandwidth[3] of realizations of $\boldsymbol{A}$ being 2. The only undirected graph with bandwidth 2 is the path graph. In bandwidth form (i.e., with a bandwidth labeling) $P\boldsymbol{B}(S) = \boldsymbol{B}(S)$ if and only if i is either end node. ■

On the other extreme, the following proposition indicates that there is only one graph controllable from all but one node.

Proposition 9. *The only connected self-damped undirected network strongly s-controllable requiring $n - 1$ inputs to be strongly s-controllable is the complete graph.*

Proof: If $n - 1$ inputs are required, by Theorem 5, the largest constrained matching in $\boldsymbol{A}$ is a 1-matching. If the network is not a complete graph then there exists some edge $\{i, j\} \notin E$, where because it is self-damped, $i \neq j$. As the network is connected then there exists some edge $\{i, p\} \in E$ with $i \neq p$, similarly there exists some edge $\{j, q\} \in E$ with $j \neq q$. Consequently, there is a constrained V_s-less 2-matching $\left\{\left\{v_i^+, v_p^-\right\}, \left\{v_q^+, v_j^-\right\}\right\}$. This satisfies Theorem 5, for $m = 2$, i.e., the network is strongly s-controllable from $n - 2$ inputs. Further, it can be shown that the largest constrained matching in $\boldsymbol{A}$ for a complete graph is a V_s-less 1-matching. From Theorem 5, the proposition follows. ■

Due to the importance of connected self-damped undirected networks, Algorithm 1 was exercised on every self-damped undirected network on 2 to 10 nodes. By testing every permutation of inputs the cardinality of the smallest strongly s-controllable inputs $|S|$ were found. The results are featured in Figure 2, where $n_D := |S|/n$. We note on average that approximately half the nodes are required to be strongly s-controllable.

To investigate the distribution of n_D across more general graphs, we examine the family of directed Erdős-Rényi random graphs.[4] Algorithm 1 was run on 20 node Erdős-Rényi graphs testing all input permutations, for samplings of $\langle k \rangle$ from 2 to 20 with each $\langle k \rangle$ sampled 1200 times. Figure 3 summarizes these results.

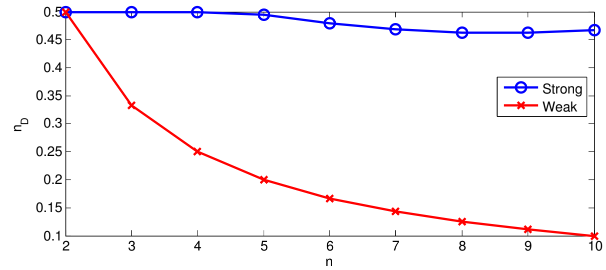

Figure 2. The variable n_D for weak and strong s-controllability for all self-damped undirected graphs for 2 to 10 nodes.

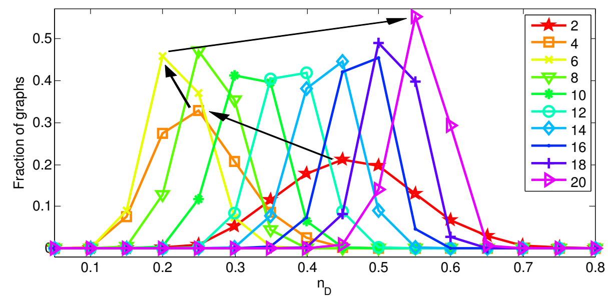

Figure 3. Directed Erdős-Rényi random graphs on 20 nodes. Each $\langle k \rangle$ value denoted in the legend was sampled 1200 times. The black arrows indicate the direction of increasing $\langle k \rangle$ values.

We observe a critical threshold phenomenon as $\langle k \rangle$ increases. Specifically for $\langle k \rangle \leq 6$, n_D decreases with increasing $\langle k \rangle$ while for $\langle k \rangle \geq 6$, n_D increases with increasing $\langle k \rangle$. Such a thresholding phenomenon is not uncommon in random networks [15]. This particular threshold can be attributed to the fact $k_c = 2\log n \approx 5.99$ is a sharp threshold for the connectedness[5] of Erdős-Rényi random networks [15], i.e., for $k < k_c$ ($k > k_c$), the network is almost surely disconnected (connected). Thus for disconnected networks, as connectivity increases, the network becomes controllable from less inputs. Further, for connected graphs as the number of edges increases n_D increases as it becomes more challenging to establish a constrained matching. From a design perspective, $k = k_c$ presents an ideal directed Erdős-Rényi random networks for on average the minimum required number of inputs for strong s-controllability.

An attraction of weak and strong s-controllability is that they provide lower and upper bounds, respectively, on the minimum number of inputs required for general controllability. Consequently, it is a fruitful exercise to compare n_D for weak and strong s-controllability on the two families of graphs presented in this section.

As there always exists an n-matching involving the self-damped edges of the network, by Theorem 1, the network

[2]A matrix is unreduced upper-Hessenberg if all entries on the first superdiagonal nonzero and all entries above this diagonal are zero.

[3]The bandwidth of a graph is the minimum $\max\{|i-j| \mid \{i,j\} \in E\}$ over all labeling of the nodes.

[4]Directed Erdős-Rényi random graphs are randomly generated graphs with an edge $(i, j) \in E$ independently existing with probability p [15]. The mean degree is defined as $\langle k \rangle = 2np$.

[5]A directed graph is connected if its underlying undirected graph is connected.

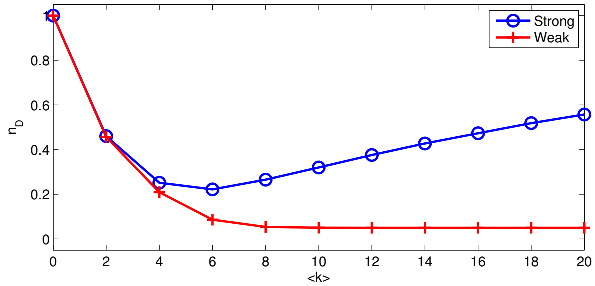

Figure 4. The variable n_D for weak and strong s-controllability for a sampling of directed Erdős-Rényi random networks on 20 nodes. Each $\langle k\rangle$ value for strong s-controllability was sampled 1200 times.

is weakly s-controllable from any arbitrary single node. Figure 2 compares the average n_D for weak and strong s-controllability. Though the weak s-controllability result implies that almost all graphs are controllable from a single node, we note that, from the strong s-controllability bound, on average the worst case on some graphs can require far more inputs.

Liu *et al.* [7] examined connected directed Erdős-Rényi random networks finding as n tends to infinity that $n_D \approx e^{-\langle k\rangle/2}$. Figure 4 compares the sampled average n_D for weak and strong s-controllability. We observe for small values of $\langle k\rangle$ that the weak and strong bounds are close but as $\langle k\rangle$ increases, n_D for weak s-controllability tends to 0 and n_D for strong s-controllability appears to tend towards 1. This is not surprising and fundamental to the difference between weak and strong s-controllability, in that weak s-controllability requires the existence of t-matchings while strong s-controllability requires constrained t-matchings. The addition of edges in a bipartite graph, for example as $\langle k\rangle$ increases, promotes t-matchings while discourages constrained t-matchings, separating the bounds.

VI. Finding Strongly S-Controllable Inputs

Validating that a given set of inputs is controllable is computationally distinct from searching for a minimum strongly s-controllable input set. This section focuses on this search problem.

The following theorem provides a bipartite based approach to find strongly s-controllable inputs.

Theorem 10. *Given a state matrix $\boldsymbol{A}$ with the constrained $(n-m_1)-$matching with unmatched nodes S_1^+ and S_1^-. Let $\boldsymbol{A}_\times$ contain a constrained V_s-less $(n-m_2)-$matching with unmatched nodes S_2^+ and S_2^-. Then the inputs associated with $S_1^- \bigcup S_2^-$, namely $S_1 \bigcup S_2$, is a strongly s-controllable input set with cardinality $m = |S_1 \bigcup S_2| \leq \min(m_1+m_2, n)$.*

Proof: First, from Proposition 7, both $\boldsymbol{A}$ and $\boldsymbol{A}_\times$ can not have constrained n-matchings, where the matching on $\boldsymbol{A}_\times$ is V_s-less. Hence, $S_1^- \bigcup S_2^-$ is nonempty. From the first condition of the theorem, $\boldsymbol{A}(S_1|\cdot)$ has a constrained $(n-m_1)$-matching. As constrained matchings are hereditary, $\boldsymbol{A}(S_1 \bigcup S_2|\cdot)$ has a constrained $(n-m)$-matching. Similarly, from the second condition of the theorem, $\boldsymbol{A}_\times(S_2|\cdot)$ has a V_s-less $(n-m_2)$-matching and consequently, $\boldsymbol{A}_\times(S_1 \bigcup S_2|\cdot)$ has a constrained V_s-less $(n-m)$-matching. Thus, from Theorem 5 the theorem follows. ∎

The smallest cardinality strongly s-controllable input set is the smallest cardinality set $S_1 \bigcup S_2$ that satisfies Theorem 10.

For the case where $\boldsymbol{A}$ has all diagonal elements nonzero, i.e., the network is self-damped, and the case where $\boldsymbol{A}$ has all diagonal elements zero, i.e., no node is self-damped, only one of the conditions in Theorem 10 needs to be validated. This result is summarized in the following corollary.

Corollary 11. *Given a state matrix $\boldsymbol{A}$ with diagonal elements all nonzero or all zero. Consider the maximum constrained self-less $(n-m)$-matching of $\boldsymbol{A}_\times$, with unmatched nodes S^+ and S^-. Then the m inputs associated with S^- is a minimum cardinality strongly s-controllable input set.*

Proof: When $\boldsymbol{A}$ has all diagonal elements nonzero, the pattern matrices $\boldsymbol{A}$ and $\boldsymbol{A}_\times$ are equal and share a maximum constrained self-less $(n-m)$-matching. When $\boldsymbol{A}$ has only zero diagonal elements then it can be formed from $\boldsymbol{A}_\times$ by removing its diagonal elements. A constrained t-matching is invariant to the removal of edges from a bipartite graph that are not members of the matching. It follows that $\boldsymbol{A}$ and $\boldsymbol{A}_\times$ share the same constrained self-less $(n-m)$-matching. From Theorem 10, the corollary follows. ∎

From Theorem 10 and Corollary 11, in the best case the problem of finding the minimum cardinality strongly s-controllable input set is equivalent to finding a maximum constrained matching. This is computationally hard; Golumbic *et al.* showed that in fact finding a maximum constrained matching in bipartite graphs is NP-complete [13]. Further, Mishra has recently shown that even approximating a maximum constrained matching is hard demonstrating a bipartite graph that can not be approximated with a factor of $\frac{1}{2\sqrt[3]{9}}n^{\frac{1}{3}-\epsilon}$ for any $\epsilon > 0$, in polynomial time [16].

We provide a greedy Algorithm 1 to produce an input set that validates each condition in Theorem 10. The algorithm involves adding control inputs at points in the algorithm that would traditionally terminate the algorithm returning an "uncontrollable" result. Applying the algorithm involves first inputting the pair $(\boldsymbol{A}, \boldsymbol{B}(S_1))$ where S_1 and T are empty. The algorithm will return an input set S_2 which validates the first condition in Theorem 10. Subsequently, applying the algorithm to the pair $(\boldsymbol{A}_\times, \boldsymbol{B}(S_2))$ and $T = V_s$ provides a set S_3 which in addition to S_2 validates the second condition in Theorem 10. Consequently, a strongly s-controllable pair is $(\boldsymbol{A}, \boldsymbol{B}(S_2 \bigcup S_3))$. Due to the challenges of establishing theoretic guarantee on the cardinality of maximum constrained matching approximation, the following section provides Monte Carlo support for the Algorithm 1's performance.

A. Algorithm's Performance

This section compares the cardinality of the strongly s-controllable input set $|S_{alg}|$ found by Algorithm 1 to the cardinality of the optimal strongly s-controllable inputs set

Algorithm 1 checks if given $\boldsymbol{A}$, $\boldsymbol{B}(S_1)$ and T, if $\boldsymbol{A}(S_1|\cdot)$ has a constrained T-less $(n-m)$-matching, where $m = |S_1|$. If not, inputs S_2 are provided to guarantee $\boldsymbol{A}(S_1 \bigcup S_2|\cdot)$ has a constrained T-less $(n-m-|S_2|)$-matching.

Initialize:
$i = n$, $j = n + m$
Replace the elements in $\boldsymbol{A}$ corresponding to members of T with the element $\otimes$
Create the row vector σ equal to the number of nonzeros, and $\otimes$'s in each column of $[\boldsymbol{A}, \boldsymbol{B}]$
Create the row vector $\sigma_\otimes$ equal to the number of $\otimes$'s in each column of $[\boldsymbol{A}, \boldsymbol{B}]$
Create the empty list S
Create the $n \times 1$ column vector p with $p(i) = i$
while $i > 0$ **do**
  Find the minimum positive value ν with column index j_s of the $1 \times j$ submatrix of σ such that $\sigma(k) = 1$ and $\sigma_\otimes(k) = 1$
  if *No such value exists* **then**
    *No such matching - Not strongly s-controllable.**
    Add $1, \ldots, i$ to S
    break
  if $\sigma_\otimes(j_s) \neq 1$ **then**
    Find the first nonzero row i_s of the column vector j_s of the $i \times j$ submatrix of $[\boldsymbol{A}, \boldsymbol{B}]$
  else
    Find the first $\otimes$ row i_s of the column vector j_s of the $i \times j$ submatrix of $[\boldsymbol{A}, \boldsymbol{B}]$
  if $i_s \neq i$ **then**
    Permute row i_s and i of the matrix $[\boldsymbol{A}, \boldsymbol{B}]$, and the column vector p
  if $j_s \neq j$ **then**
    Permute the column j_s and j of the matrix $[\boldsymbol{A}, \boldsymbol{B}]$, and the row vectors σ and $\sigma_\otimes$
  if $\nu = 1$ **then**
    $j = j - 1$
  else
    *No such matching - Not strongly s-controllable.**
    Add i to S
  foreach $k = 1, \ldots, j$ **do**
    if $[\boldsymbol{A}, \boldsymbol{B}]_{ik}$ *is* $\otimes$ **then**
      $\sigma_\otimes(k) = \sigma_\otimes(k) - 1$
      $\sigma(k) = \sigma(k) - 1$
    else if $[\boldsymbol{A}, \boldsymbol{B}]_{ik}$ *is nonzero* **then**
      $\sigma(k) = \sigma(k) - 1$
  $i = i - 1$
if S *is empty* **then**
  Found such a matching.
else
  Construct the set $S_2 = \{p(S(i))|i = 1, \ldots, |S|\}$

$|S_{opt}|$, found through an exhaustive check of all input permutations.

For each of the self-damped undirected graphs up to 10 nodes, the ratio $R = |S_{alg}| \,/\, |S_{opt}|$ was calculated. All such graphs were within a factor of 2 of the optimal cardinality. Similarly, R was calculated for each Erdős-Rényi random networks on $\langle k \rangle = \{2, 6, 10, 14, 18\}$, and summarized in Table I. All such graphs were within a factor of 2.3 of the optimal cardinality. The poorest performance occurred about the threshold value $\langle k \rangle = 6$.

VII. Conclusion

This paper presents an analysis of strong structural controllability in networked systems. We provided an equivalent constrained matching condition for strong structural controllability. A polynomial time algorithm to validate these conditions and to form strong structural controllable input sets was then presented. We proceeded to show that the search for a minimum cardinality strong structural controllability input set is NP-complete; finding a factor approximation is also shown to be difficult. Insights were provided into the spread of minimum controllable input sets using weak and strong structural controllability bounds. Future work of particular interest involves establishing conditions when weak and strong structural controllability share similar size minimum cardinality input sets.

Table I
Fraction of $\langle k \rangle$ Erdős-Rényi random networks on 20 nodes that exhibit an R value less than 1.0, 1.3, 1.7, 2.0 and 2.3, respectively. Here, $\epsilon = 0.0001$.

		R				
		1.0	1.3	1.7	2.0	2.3
$\langle k \rangle$	2	0.54	0.99	1.00	1.00	1.00
	6	0.18	0.61	0.90	$1-40\epsilon$	1.00
	10	0.26	0.90	$1-30\epsilon$	1.00	1.00
	14	0.34	0.97	1.00	1.00	1.00
	18	0.39	$1-8\epsilon$	1.00	1.00	1.00

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