Linear functionals & Adjoints

Linear functions $L(V, \mathbb{F}) = \mathbb{F}^*$

dual space.

$V^*$ is a vector space.

Ex: Let $V$ be the set of continuous functions on $[a, b]$
Then $\int_a^b f(x) \, dx$, $(f \in V)$ is a linear functional on $V$.

Ex: Let $u_1, \ldots, u_n$ be basis for $V$. Consider the expansion of

$\mathbf{v} = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$

$\alpha_i$'s depend on $\mathbf{v}$, i.e., $\alpha_i(\mathbf{v})$

$\alpha_i : V \to \mathbb{F}$ is a linear functional.
A notation: $(v, v^*)$ means the operation of applying $v^*$ on $v$, e.g., $v^*(v)$.

Now if we write

$$v = \sum \lambda_i v_i + \ldots + \alpha_n v_n.$$  

The $v_i^*$'s form a basis for $V^*$ if all functionals on $V$ are linear.

Consider $v^* \in V^*$.

We want to show that $v^* = \sum \beta_j v_j^*$ for some $\beta_j$'s.

Let $v \in V$. Then

$$(v, v^*) = \left( \sum \lambda_i (v, v_i^*) v_i, v^* \right) = \sum \lambda_i (v, v_i^*) (v_i, v^*)$$

$v^*$ is a linear functional!
\[
\begin{aligned}
&= (v, \sum (v_j^* u_j^*) v_j^*) \\
\Rightarrow &\quad u^* = \sum \beta_j v_j^* \\
&\text{Note that if } (v, u_1^*) = (v, u_2^*) \neq 0 \quad \forall v \\
&\Rightarrow (v, u_1^* - u_2^*) = 0 \quad \forall v \\
&\Rightarrow u_1^* = u_2^*.
\end{aligned}
\]

(independence)

Assume that \exists \beta_j's, not all zero s.t.

\[
\sum \beta_j v_j^* = 0.
\]

But

\[
\beta_k = (v_k, \sum \beta_j v_j^*) = 0 \quad \forall k.
\]
So if \( \{ v_1, \ldots, v_n \} \) basis for \( V \),
\[ \{ v_1^*, \ldots, v_n^* \} \text{ basis for } V^* \]

\( v_i^* \) coefficient of vectors in \( V \)
along \( v_i \)

\[ \dim V = \dim V^* \]

& \( (v_j, v_k^*) = 0 \quad \text{if} \quad k \neq j \)

\( (v_j, v_j^*) = 1 \)

So for finite dimensional vector space there is an isomorphism
\[ \phi : V \rightarrow V^* \]
Suppose \( \mathbf{V} \) is finite dimensional over \( \mathbb{R} \).

Then \( \mathbf{V} \) is isomorphic to \( \mathbb{R}^n \).

Choose the standard basis:

\[ \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n. \]

\[ \mathbf{v} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + \cdots + v_n \mathbf{e}_n. \]

\[ \mathbf{v} \in \mathbf{V}. \]

\[ (v, v_i^*) = (\mathbf{v}, \mathbf{v}_i^*) = \sum_{j=1}^{n} \beta_j (v, v_j^*) = \sum_{j=1}^{n} \beta_j v_j. \]

\[ \mathbf{v}^T \mathbf{v} = \beta^T \mathbf{v} = \beta^T \mathbf{v} \in \mathbb{R}^n. \]
So any linear functional on $\mathbb{R}^n$ can be represented as $c^T v$ for some $c \in \mathbb{R}^n$.

Observation

In fact, this has a natural generalization in terms of an inner product on arbitrary vector spaces (on reals):

$$\langle \cdot , \cdot \rangle : V \times V \rightarrow \mathbb{R}$$

(see chp. 6)

S.t.

\[
\begin{cases}
\langle u, v \rangle \text{ is bilinear (linear in each argument)} \\
\langle u, v \rangle = \langle v, u \rangle \quad \forall u, v \in V \\
\langle u, v \rangle \geq 0 \quad \forall u, v \in V \\
\langle u, v \rangle = 0 \iff u = 0.
\end{cases}
\]
Also notice that if $V = \mathbb{R}^n$ (or $\mathbb{R}$)
then $V^*$ is a copy of $\mathbb{R}^n$.

The "c" in "c*" in $\mathbb{R}^n$ is the dual of the dual.

We can in fact say that in general

$$(V^*)^* = V$$

the dual of the dual.
Annihilator:

We will see later on that the inner product can be used in spaces (that have an inner product) for example by saying that $(a, b) = 0$ means that $a \perp b$. But we can in fact think about this without introducing an inner product as well:

Let $A \subseteq W$

then annihilator of $A$ is

$$A^\perp = \{ v^* \in W^* \mid (v, v^*) = 0 \; \forall v \in W \}$$
$A$ is a subspace of $V^*$ s.t. if $v_1, v_2 \in A$ then $\forall v \in A$

$$
\begin{align*}
(v_1, v_1^*)&= 0 \\
(v_2, v_2^*) &= 0
\end{align*}
$$

\Rightarrow \quad (v, \alpha v_1^* + \beta v_2^*) = 0 \quad \forall v \in A

If $A$ is a subspace in $V$ with basis $v_1, \ldots, v_k$ then

$$
A^* = \{ v^* \in V^* \mid (v_i, v^*) = 0 \quad \forall i = 1, \ldots, k \}
$$
Theorem: Let $W \subseteq V$ be a subspace. Then
\[ \dim W + \dim W^\perp = \dim V. \]

Proof:

- Basis $w_1, \ldots, w_k, v_1, \ldots, v_n.
- \text{Reorder } v_i's \text{ such that } w_1, \ldots, w_k, v_{k+1}, \ldots, v_n \text{ is a basis for } V.

Claim: $v_{k+1}, \ldots, v_n$ is a basis for $W^\perp$.

Notice that $(w_i, v_l^*) = 0$ when $i = 1, \ldots, k$ and $l = k+1, \ldots, n$.

\[ \Rightarrow \text{span } \{v_{k+1}^*, v_{k+2}^*, \ldots, v_n^*\} \subseteq W^\perp \text{ is independent.} \]
Suppose \( w^* \in W^* \) \( \Rightarrow \) \( w^* = \sum_{j=1}^{n} \alpha_j v_j^* \)

But \( (v_i^*, w^*) = (v_i^*, \sum_{j=1}^{n} \alpha_j v_j^*) = \sum_{j=1}^{n} \alpha_j (v_i^*, v_j^*) \) 

\( = \alpha_i \) 

So if \( w^* \in W^* \downarrow \)

\( (v_1^*, w^*) = 0 \\
(v_2^*, w^*) = 0 \\
\vdots \\
(v_k^*, w^*) = 0 \)

\( \Rightarrow w^* = \sum_{j=k+1}^{n} \alpha_j v_j^* \)

So \( W^* = \text{span} \{ v_{k+1}^*, \ldots, v_n^* \} \)
We can compose

\[ w^* T(v) = w^*(Tv) = (Tv, w^*) \]

by a linear functional on \( \mathfrak{F} V \)
So:

\[ T^* : \mathcal{V}^* \rightarrow \mathcal{W}^* \quad \text{defined through } T \]

\[ \rightarrow \text{adjoint of } T \]

The defining property of the adjoint is that

\[ (Tv, w^*) = (v, T^*w^*) \]
Let us look at $\mathbb{R}^n$, $\mathbb{R}^m$ for intuition.

$A \in \mathbb{R}^{m \times n}$

$\mathbb{R}^n \xrightarrow{w^*} (\mathbb{R}^n)^* \xrightarrow{A^*} (\mathbb{R}^m)^* \xrightarrow{} \mathbb{R}^m$

$(w^*, Av) = (v, A^*w^*)$

$w^T(Av) = v^T A^* w^*$

$A^* = A^T$!
Prop.

Let \( T \in \mathbb{L}(V, W) \) & \( T^* \) is the adjoint of \( T \).

Then

\[ R(T) \perp = \ker(T^*) \]

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Proof: Suppose \( w^* \in R(T) \perp \)

\[ (Tw, w^*) = 0 \quad \forall \ v \in V \]

\[ (v, T^* w^*) = 0 \quad \forall \ v \in V \]

\[ \Rightarrow T^* w^* = 0 \iff w^* \in \ker(T^*) \]
Similarly to show

\[ R(T^*)^\perp = \ker(T) \]

we can start with \( T^* \) and apply the first part

by noticing that \( T^{**} = T \).

Or \( v \in R(T^*)^\perp \)

Then \( (T^{\ast\ast} w^*, v) = 0 \) \( \forall w^* \in W^* \)

\( (w^*, T v) = 0 \) \( \forall w^* \in W^* \)

\( \Rightarrow T v = 0 \) \( \Rightarrow v \in \ker(T) \)
When a linear operator has a property that the adjoint of the operator is the same as the operator, the operator is called *self-adjoint*.

mostly used in the context of inner products.