Once we fix $V$ & $W$, we can consider the "set" of all linear operators from $V$ to $W$. Denote this set $\mathcal{L}(V, W)$.

Guess what? This set is a vector space!

$$S, T \in \mathcal{L}(V, W)$$

$$\begin{cases}
S + T \in \mathcal{L}(V, W) \\
\alpha S \in \mathcal{L}(V, W)
\end{cases}$$

When $W = F \rightarrow \mathcal{L}(V, W) = V^*$

When $W = V \rightarrow \mathcal{L}(V, V) = \mathcal{L}(V)$

The members of $V^*$ are called linear functionals on $V$. 

Dual space:
Consider $L(V)$, the vector space of all linear operators from $V$ to $V$.

Then we can define "multiplication" between elements in $L(V)$ by using composition:

Let $S, T \in L(V)$

Then

$$ST(v) = S(Tv).$$

Note that $ST \in L(V)$ if $S, T \in L(V)$

$$ST(\alpha v + \beta w) = \alpha STv + \beta STw.$$
Moreover we can show that

\[ R(ST) = (RS)T \] \{ associative \}

\[ R(S+T) = RS + RT \] \{ distributive \}

\[ (R+S)T = RT + ST \]

So we have

\[ \alpha \quad \circ \quad \alpha' \quad T \]

\[ S + T \]

\[ \text{distributive/associative} \]

\[ L(V) \]

\[ \text{invertible linear operators} \]
This is called an algebra.

(Note that multiplication is not necessarily commutative.)

In this algebra there is a subset $S$ such that:
- The elements have an inverse.

$GL(V) = \text{general linear group on } V$

\[
\downarrow
\]

\[ = \{ S \in L(V) \mid S \text{ is invertible} \}\]

A group in the algebra $L(V)$, $S$ such that:

$ST = TS, S = I$
Def: \( S, T \in \text{GL}(V) \) are called conjugates if \( \exists R \in \text{GL}(V) \) s.t.

\[
T = RSR^{-1}
\]

When we specialize this to \( V = \mathbb{R}^n \), then we call two matrices \( A, B \in \mathbb{R}^{n \times n} \) conjugates (or similar) if \( \exists C \in \text{GL}(\mathbb{R}^n) \) s.t.

\[
A = CBC^T
\]
In the case of matrices $AB$ has a nice interpretation:

$$
AB = A \begin{bmatrix}
    b_1 \\
    b_2 \\
    \vdots \\
    b_p
\end{bmatrix}
\begin{bmatrix}
    a_1^T \\
    a_2^T \\
    \vdots \\
    a_m^T
\end{bmatrix} = \left[ A b_1, A b_2, \ldots, A b_p \right]_{m \times p}.
$$

Columns of $AB$ are linear combinations of columns of $A$, $b$:

$$
AB = \begin{bmatrix}
    a_1^T \\
    a_2^T \\
    \vdots \\
    a_m^T
\end{bmatrix} B = \begin{bmatrix}
    a_1^T B \\
    a_2^T B \\
    \vdots \\
    a_m^T B
\end{bmatrix}
$$
What is \( a_i^T B = [x_1 \ldots x_n] \) 

\[
= \sum_i x_i r_i(B) \quad \text{linear combination of rows of } B.
\]

Thus: if \( A \in \mathbb{R}^{n \times n} \) is invertible then \( r(A) = n \).

If \( A \in \text{GL}(n) \), then \( \exists B \) s.t. \( \begin{array}{c} \text{BA = I} \\ \text{row rank = n} \end{array} \)

\( \#	ext{ of linearly independent columns (or rows) of } A \).

Row rank is not bigger than row rank of \( A \) \( \Rightarrow \) row rank of \( A \) is at least \( n \)!

(can't be bigger than \( n \) either!)
Suppose we are given a linear operator $T$ on the finite dimensional vector space $T \in \mathcal{L}(W, W)$

$$T: V \rightarrow W$$

$$\{v_1, v_2, \ldots, v_m\} \rightarrow \{w_1, \ldots, w_m\}$$

Basis for $V$

Basis for $W$

Then

$$T v_i = a_{i1} w_1 + a_{i2} w_2 + \cdots + a_{im} w_m$$

$$\begin{bmatrix}
\vdots & a_{1i} & \vdots \\
\vdots & a_{2i} & \vdots \\
\vdots & a_{mi} & \vdots \\
\end{bmatrix}
= A_{T, v_i, W}$$

basis in $W$

representation of $T v_i$ in $\mathcal{L}(V, \mathcal{L}(W, W))$ via $w_1, \ldots, w_m$
So $A_T$ depends on the choice of $v$ and $w$.

Let us use the notation $C_w x$: the coordinates of $x$ using the basis $w = \{ v_1, \ldots, v_m \}$.

So if $x = a_1 v_1 + a_2 v_2 + \cdots + a_m v_m$

Then $C_w x = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}$

$\Rightarrow C_w^T x = A_{T,v,w} C_w x$

a linear operator $T$ and the matrix representation of $T$
Let \( X \in \mathbb{V} \), a vector space over \( \mathbb{R} \).

\[ T \in \mathcal{L}(\mathbb{V}) \]

and two bases \( \mathcal{B} = \{ v_1, v_2, \ldots, v_n \} \)

\( \mathcal{B} = \{ w_1, w_2, \ldots, w_n \} \).

Then:

\[ \begin{align*}
C_v &: \mathbb{V} \rightarrow \mathbb{R}^n \\
C_w &: \mathbb{V} \rightarrow \mathbb{R}^n
\end{align*} \]

two different isomorphisms.

What is \( A_{T,v} \) in terms of \( A_{T,w} \)?

Note that

\[ \begin{array}{c}
\mathcal{C}_w \downarrow \mathbb{V} \\
\mathbb{R}^n \downarrow \downarrow \downarrow \\
\mathcal{T} \downarrow \downarrow \downarrow \\
\mathcal{C}_v \downarrow \downarrow \downarrow \\
\mathbb{R}^n \downarrow \downarrow \downarrow \\
\downarrow \downarrow \downarrow \\
A_{T,v,w,v}
\end{array} \]

\( \Leftrightarrow \) short hand for \( A_{T,w,v} \)
So maybe we need to go from $C_W$ to $C_V$.

$x \in V$

$C_V x \quad C_W x$

Taking basis $v$ to $v$

So if we have the rep. in $V$ of $x$

we can apply $C_W C_V$ to get the rep. in $W$ of $x$. 
\[ V \rightarrow V \]
\[ C_{v} \downarrow \quad R^{n} \rightarrow \quad A_{T,w} \downarrow \quad C_{v}C_{w}^{-1} \]
\[ C_{v}C_{w}^{-1} \downarrow \quad R^{n} \rightarrow \quad A_{T,v} \]

\[ A_{T,w} = C_{v} \circ A_{T,v} \circ C_{v,w} \]

OR

\[ A_{T,v} = C_{v,w} \circ A_{T,w} \circ C_{w,v} \]

a nonsingular map taking \( w \rightarrow v \) is a bijection.
Two matrices are called similar if they represent the same $T$ in different basis:

$$A_1 = A_{T,v}$$
$$A_2 = A_{T,w}$$

Thus $A_1$ and $A_2$ are similar if

$$A_1 = C^{-1} A_2 C$$

for some $C \in GL(n)$.

A similar def. can stated for linear operators $S, T \in L(V)$ are similar (or conjugate under $GL(V)$) if $\exists R \in GL(V)$ s.t.

$$T = R S R^{-1}$$
Recall that \( R^{m \times n} \):

\[
\begin{align*}
\mathcal{N}(A) &= \{ x \in \mathbb{R}^n \mid Ax = 0 \} \\
\mathcal{R}(A) &= \{ Ax \mid x \in \mathbb{R}^n \}
\end{align*}
\]

for a linear operator, we can define analogous concepts:

\[
T : V \to W , \quad T \in \mathcal{L}(V,W)
\]

\[
\ker(T) = \{ v \in V \mid Tv = 0 \}
\]

\[
\text{Range}(T) = \{ w \in W \mid w = Tv \text{ for some } v \in V \}
\]

We have seen that \( \mathcal{N}(A) \) & \( \mathcal{R}(A) \) are subspaces of \( \mathbb{R}^n \) & \( \mathbb{R}^m \) respectively—similar statements hold true for linear operators.
\[ f(T) = \text{rank of the linear operator } T \]
\[ = \text{dimension of } \text{Range}(T) \]
\[ V(T) = \text{dimension of } \text{Ker}(T) \]
\[ = \text{nullity of the linear operator } T. \]

Then: let \( V \) be a finite-dimensional vector space
& \( W \) be an arbitrary vector space &
\( T \in \mathcal{L}(W, W) \).

Then
\[ f(T) + V(T) = \dim V \]
Proof: Let \( \dim V = n \).

Let \( \text{Ker}(T) \) admit the basis \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_l \) and linearly independent in \( V \).

Add \( \mathbf{v}_{l+1}, \mathbf{v}_{l+2}, \ldots, \mathbf{v}_n \) from the basis in \( V \).

So \( \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_l, \mathbf{v}_{l+1}, \mathbf{v}_{l+2}, \ldots, \mathbf{v}_n \} \) is a basis for \( V \).

Is it true that \( k = \text{rank}(T) \)?

\[ \Rightarrow \dim \text{Range}(T) \]

Take \( \mathbf{w} \in \text{Range}(T) : \mathbf{w} = T(\mathbf{v}) \) for some \( \mathbf{v} \in V \).

\[ \mathbf{w} = \sum_{i=1}^{l} \alpha_i \mathbf{v}_i + \sum_{j=l+1}^{n} \beta_j \mathbf{v}_j \]

\[ \Rightarrow \mathbf{w} = T \left( \sum_{i=1}^{l} \alpha_i \mathbf{v}_i + \sum_{j=l+1}^{n} \beta_j \mathbf{v}_j \right) = \sum_{i=0}^{n} \beta_j \mathbf{T} \mathbf{v}_j \]
So \( \{ T u_{l+1}, T u_{l+2}, \ldots, T u_n \} \) span \( \text{Range}(T) \).

What about independence?

Suppose \( \exists \ c_i \) is not all zero s.t.

\[
\sum_{i=l+1}^{n} c_i T u_i = T \left( \sum_{i=l+1}^{n} c_i u_i \right) = 0.
\]

\[
\Rightarrow \sum_{i=l+1}^{n} c_i u_i \in \text{Ker}(T) \quad \text{not all } c_i \text{'s zero}.
\]

But this says that

\[
\sum_{i=l+1}^{n} c_i u_i = \sum_{j=1}^{l} d_j v_j
\]

\[
\Rightarrow \sum_{j=1}^{l} d_j v_j - \sum_{i=l+1}^{n} c_i u_i = 0 \quad \Rightarrow \quad \text{since } v_j \text{'s & } u_i \text{'s are linearly independent}.
\]

\[
\text{not all } c_i \text{'s & } d_j \text{'s zero}.
\]
Note that if

\[ v_1 - v_2 \in \text{Ker}(T). \]

Then \[ T(v_1 - v_2) = 0 \Rightarrow TV_1 = TV_2. \]

So \( T \) assigns the same value to \( v_1 \)!

\[ T : W \rightarrow W \]

can be written as:

\[ T : W \rightarrow W/\text{Ker}(T) \]

\[ \dim W/\text{Ker}(T) \rightarrow T(W). \]

\[ \dim W/\text{Ker}(T) = \dim \text{of } T(W). \]

\[ n - \nu(T) \rightarrow \chi(T). \]
This means that if $T \in \mathcal{L}(V)$ is finite dim.

\[ \nu(T) = 0 \Rightarrow \dim \ker T = \dim \operatorname{range} T = 0 \]

\[ \Rightarrow V = \operatorname{Range}(T) \]

The map $T$ is bijective & invertible.