Stability of Nonlinear Networks via M-matrix Theory: Beyond Linear Consensus

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Motivation: Extending Linear Consensus

Consensus Model

\[
\dot{x}_i(t) = \sum_{(j,i) \in E} w_{ij} (x_j(t) - x_i(t)) \iff \dot{x}(t) = -L(G)x(t)
\]

Example: Load balancing to minimize completion time over homogenous linear machines
Motivation: Extending Linear Consensus

Nonlinear Output Consensus Model

\[ \dot{x}_i(t) = \sum_{(j,i) \in E} w_{ij} (y_j(t) - y_i(t)) \iff \dot{x}(t) = -L(G)y(t) \]

\[ y_i(t) = f_i(x_i(t)) \quad y(t) = f(x(t)) \]

Example: Load balancing to minimize completion time over heterogenous nonlinear machines

Applications: Social, chemical, biological networks

Is there a class of networked systems of the form \( \dot{x} = -A(G)f(x) \) that are stable?
Outline

- Related Research
- Formal Problem Statement
- Background
- Stability Analysis
- Characteristics of the Equilibrium set
- Extension
- Conclusion
Related Research

- **M-matrices**: Nonnegative matrices, Z-matrices (Berman and Plemmons 1979)
  - Population migration, Markov processes, economic systems, discretized of differential operators

- **Stability of Nonlinear Systems over Networks**
  - Siljak 1970s, convergence to a single equilibrium
  - Araki and Kondo 1972, decomposed networks into subsystems and provided DC gain conditions
  - Xiang and Chen 2007, passivity measures of individual node’s stability

- **Nonlinear Consensus**
  - Cortes 2008; Hui et al. 2008, general conditions to achieve nonlinear consensus
  - Yu et al. 2011, $\dot{x}_i = f(x_i) - c \sum_{j=1}^{n} L_{ij} \Gamma x_j(t)$
  - Ajorlou et al. 2010, $\dot{x} = \sum_{(i,j) \in E} f(x_i - x_j)$

- **Output Consensus**: $z = -Ay$ provided and $\lim_{t \to \infty} (y_i - y_j) = 0$
  - Yang et al. 2011; Kim et al. 2011; Vengertsev et al. 2010, $\dot{x} = Ax + Bu,$ $y = Cx$
  - Yumei et al. 2011, $y_i = f_i(x_i)$, common strictly increasing $f_i(x)$ over undirected graphs
Problem Statement

- Establish convergence for

\[ \dot{x}_i(t) = -w_{ii}f_i(x_i(t)) + \sum_{(j,i) \in E} w_{ij}f_j(x_j(t)) \]

\[ \dot{x}(t) = -A(G)y(t) \text{ where } y_i(t) = f_i(x_i(t)) \]

to the set \( A = \{ x \in \mathbb{R}^n | A(G)f(x) = 0 \} \)

Assumptions:
- Self loops \( w_{ii} \)'s are sufficiently large so that \( 0 \leq \text{Re}(\lambda_1(A)) \leq \text{Re}(\lambda_2(A)) \leq \ldots \)
- \( G \) is strongly connected
  - \( \iff A(G) \) is an irreducible M-matrix (\( a_{ij} \leq 0 \) and \( \text{Re}(\lambda_i(A)) \geq 0 \))
- \( f_i(\cdot) : \mathbb{R} \rightarrow \mathbb{R} \) is continuous and radially unbounded
  - \( \int_0^{x_i} f_i(y)dy \rightarrow \infty \text{ as } |x_i| \rightarrow \infty \)
For nonsingular $A$

- $\mathcal{A} = \{x \in \mathbb{R}^n : y_i = f_i(x_i) = 0\}$, i.e., the roots of $f_i(x)$

For singular $Av = 0$, $w^T A = 0$ then $v_i, w_i > 0$

- $\mathcal{A} = \{x \in \mathbb{R}^n : v_1 y_1 = v_2 y_2 = \cdots = v_n y_n\}$
- $\sum_{i=1}^n w_i x_i$ is invariant

For $L(G)$, $v = 1$ and $w_i = \sum_{T \in \mathcal{T}_i} \prod_{e_{kj} \in T} w_{jk}$ where $\mathcal{T}_{i}$ is directed spanning trees rooted at node $i$ (for balanced $G$, $w = 1$)

- $\mathcal{A} = \{x \in \mathbb{R}^n : y_1 = y_2 = \cdots = y_n\}$
- (balanced) $\sum_{i=1}^n x_i$ is invariant

$A(G)$ is an irreducible M-matrix implies $\exists$ a positive diagonal $D$ s.t.,

$$DA + A^T D \succeq 0$$
Stability Analysis

Stability Lemma

For the model, \( x(t) \to \mathcal{A} \). If the intersection of the invariant set of the dynamics and \( \mathcal{A} \) is composed of a finite number of isolated points then \( x(t) \to x_e \) for some \( x_e \in \mathcal{A} \).

Proof Outline: \( D \) is positive diagonal matrix such that \( DA + A^T D \succeq 0 \). Consider

\[
V(x) = \sum_{i=1}^{n} [D]_{ii} \int_{0}^{x_i} f_i(z) \, dz
\]

\[
\dot{V}(x) = -\frac{1}{2} f(x)^T (DA + A^T D) f(x) \leq 0.
\]

The largest invariant set is \( \mathcal{A} \), so by LaSalle’s theorem the result follows.

Consensus Stability

For the model when \( A = L(G) \), output consensus is attained, i.e.,
\[
\lim_{t \to \infty} |y_i - y_j| = 0, \ \forall i,j.
\]
Isolated Equilibrium

- Example of non-isolated equilibriums:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = - \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -x_1 \\
x_2
\end{bmatrix} = -L(P_2) \begin{bmatrix} y_1 \\
y_2
\end{bmatrix}
\]

Isolated

The equilibrium \( x_e \in \mathcal{A} \) is isolated for nonsingular \( A \) if \( f(x) \) is differentiable at \( x_e \) and \( \frac{d}{dx_i} f_i([x_e]_i) \neq 0 \) for all \( i = 1, \ldots, n \). The equilibrium is isolated for singular \( A \) if in addition

\[
\sum_{i=1}^{n} \frac{v_i w_i}{d/dx_i f_i([x_e]_i)} \neq 0, \text{ for all } i = 1, \ldots, n.
\]

Corollary

The equilibrium \( x_e \in \mathcal{A} \) is isolated if \( f_i(x) \) is differentiable at \( x_e \) and strictly increasing for all \( i = 1, \ldots, n \).
Strictly Increasing Functions

Unique Equilibrium
If \( f_i(x) \) is differentiable and strictly increasing for all \( i = 1, \ldots, n \) then, for \( A \) nonsingular, \( x_e = f^{-1}(0) \) is the unique stable equilibrium. For \( A \) singular, \( x_e = f^{-1}(\beta v) \) is the unique equilibrium where \( \beta \in \mathbb{R} \) “uniquely” satisfies \( w^T x_e = w^T x_0 \) such that \( Av = 0 \) and \( w^T A = 0 \).

Consensus
For \( A = L(G) \), if \( f_1(x) = f_2(x) = \cdots = f_n(x) \) is differentiable and strictly increasing then \( \lim_{t \to \infty} |x_i - x_j| \to 0 \), \( \forall i, j \), i.e. internal state consensus is attained. Further if \( G \) is balanced then average internal state consensus is attained.
Extension

- Modified Dynamics with the addition of a nonlinear term,

\[
\dot{x}_i(t) = -a_{ii}f_i(x_i(t)) + \sum_{j \neq i} a_{ij}f_j(x_j(t)) - g_i(x_i)
\]

\[
\dot{x} = -Ay - g(x) \text{ where } y_i = f_i(x_i),
\]

giving an equilibrium set \( B = \{ x \in \mathbb{R}^n | Af(x) + g(x) = 0 \} \)

- Additional assumptions: \( g_i(\cdot) \in \mathbb{R} \rightarrow \mathbb{R} \) is continuous and \( f_i(x)g_i(x) \geq 0 \)

Stability Lemma

For the model, \( x(t) \rightarrow B \). If the intersection of the invariant set of the dynamics and \( B \) is composed of a finite number of isolated points then \( x(t) \rightarrow x_e \) for some \( x_e \in B \).

- Application to neural networks: where \( g_i(x_i) = \gamma_i x_i, \gamma_i > 0, f_i(\cdot) \) is increasing and \( f_i(0) = 0 \). Hence, \( B = \{ 0 \} \) and the origin is globally asymptotically stable
Established sufficient conditions for stability of a large class of nonlinear dynamic networks

Explored its equilibrium set

Discussed the results’ implications for output consensus

Extended the model with the addition of an agent dependent nonlinear term

Future work involves the introduction of control terms into the dynamics