Stability of Nonlinear Networks via M-matrix Theory: Beyond Linear Consensus

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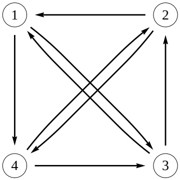
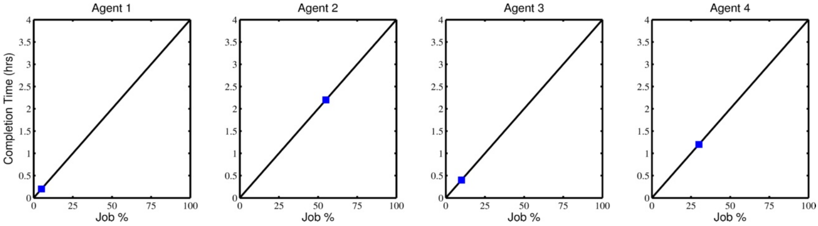
Motivation: Extending Linear Consensus

Consensus Model

$$\dot{x}_i(t) = \sum_{(j,i)\in E} w_{ij}\,(x_j(t) - x_i(t)) \iff \dot{x}(t) = -L(\mathcal{G})\,x(t)$$



Example: Load balancing to minimize completion time over homogenous linear machines

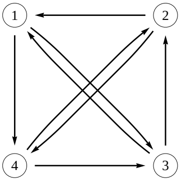


Motivation: Extending Linear Consensus

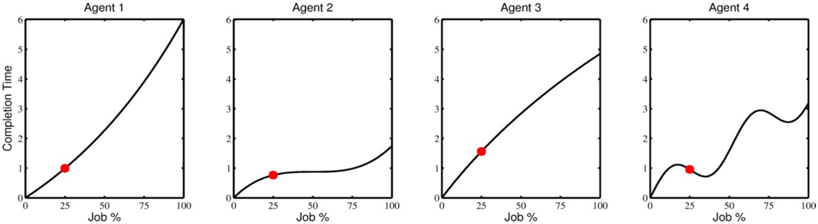
Nonlinear Output Consensus Model

$$\dot{x}_i(t) = \sum_{(j,i)\in E} w_{ij}\,(y_j(t) - y_i(t)) \iff \dot{x}(t) = -L(\mathcal{G})\,y(t)$$

$$y_i(t) = f_i(x_i(t)) \qquad\qquad y(t) = f(x(t))$$



Example: Load balancing to minimize completion time over heterogenous nonlinear machines



Applications: Social, chemical, biological networks

Is there a class of networked systems of the form $\dot{x} = -A(\mathcal{G})f(x)$ that are stable?

Outline

- Related Research
- Formal Problem Statement
- Background
- Stability Analysis
- Characteristics of the Equilbrium set
- Extension
- Conclusion

Related Research

- M-matrices: Nonnegative matrices, Z-matrices (Berman and Plemmons 1979)
  - Population migration, Markov processes, economic systems, discretized of differential operators
- Stability of Nonlinear Systems over Networks
  - Siljak 1970s, convergence to a single equilibrium
  - Araki and Kondo 1972, decomposed networks into subsystems and provided DC gain conditions
  - Xiang and Chen 2007, passivity measures of individual node's stability
- Nonlinear Consensus
  - Cortes 2008; Hui *et al.* 2008, general conditions to achieve nonlinear consensus
  - Yu *et al.* 2011, $\dot{x}_i = f(x_i) - c\sum_{j=1}^{n} L_{ij}\Gamma x_j(t)$
  - Ajorlou *et al.* 2010, $\dot{x} = \sum_{(i,j)\in E} f(x_i - x_j)$
- Output Consensus: $z = -Ay$ provided and $\lim_{t\to\infty}(y_i - y_j) = 0$
  - Yang *et al.* 2011; Kim *et al.* 2011; Vengertsev et al. 2010, $\dot{x} = Ax + Bu$, $y = Cx$
  - Yumei et al. 2011, $y_i = f_i(x_i)$, common strictly increasing $f_i(x)$ over undirected graphs

Problem Statement

- Establish convergence for

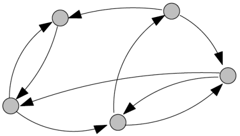
$$\dot{x}_i(t) = -w_{ii} f_i(x_i(t)) + \sum_{(j,i)\in E} w_{ij} f_j(x_j(t))$$

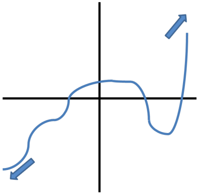
$$\dot{x}(t) = -A(\mathcal{G})y(t) \text{ where } y_i(t) = f_i(x_i(t))$$

  to the set $\mathcal{A} = \{x \in \mathbb{R}^n | A(\mathcal{G})f(x) = 0\}$

Assumptions:

- Self loops w_{ii}'s are sufficiently large so that $0 \leq Re(\lambda_1(A)) \leq Re(\lambda_2(A)) \leq \dots$
- $\mathcal{G}$ is strongly connected
  - $\iff A(\mathcal{G})$ is an irreducible M-matrix ($a_{ij} \leq 0$ and $Re(\lambda_i(A)) \geq 0$)
- $f_i(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and radially unbounded
  - $\int_0^{x_i} f_i(y)dy \rightarrow \infty$ as $|x_i| \rightarrow \infty$





Background

- For nonsingular A
  - $\mathcal{A} = \{x \in \mathbb{R}^n : y_i = f_i(x_i) = 0\}$, i.e., the roots of $f_i(x)$
- For singular $Av = 0$, $w^T A = 0$ then $v_i, w_i > 0$
  - $\mathcal{A} = \{x \in \mathbb{R}^n : v_1 y_1 = v_2 y_2 = \cdots = v_n y_n\}$
  - $\sum_{i=1}^n w_i x_i$ is invariant
- For $L(\mathcal{G})$, $v = \mathbf{1}$ and $w_i = \sum_{T \in \mathcal{T}_i} \prod_{e_{kj} \in T} w_{jk}$ where $\mathcal{T}_i$ is directed spanning trees rooted at node i (for balanced $\mathcal{G}$, $w = \mathbf{1}$)
  - $\mathcal{A} = \{x \in \mathbb{R}^n : y_1 = y_2 = \cdots = y_n\}$
  - (balanced) $\sum_{i=1}^n x_i$ is invariant
- $A(\mathcal{G})$ is an irreducible M-matrix implies $\exists$ a positive diagonal D s.t., $DA + A^T D \succeq 0$

Stability Analysis

Stability Lemma

For the model, $x(t) \to \mathcal{A}$. If the intersection of the invariant set of the dynamics and $\mathcal{A}$ is composed of a finite number of isolated points then $x(t) \to x_e$ for some $x_e \in \mathcal{A}$.

Proof Outline: D is positive diagonal matrix such that $DA + A^T D \succeq 0$. Consider

$$V(x) = \sum_{i=1}^{n} [D]_{ii} \int_0^{x_i} f_i(z)\, dz$$

For $f_i(x) = x$,
$V(x) = \sum_{i=1}^{n} x_i^2 = x^T x$

$$\dot{V}(x) = -\frac{1}{2} f(x)^T \left(DA + A^T D\right) f(x) \leq 0.$$

The largest invariant set is $\mathcal{A}$, so by LaSalle's theorem the result follows.

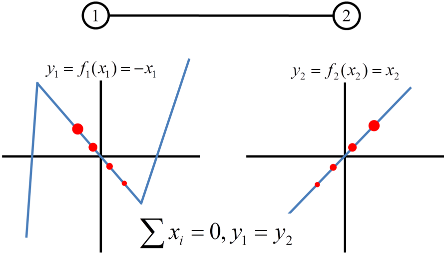
Consensus Stability

For the model when $A = L(\mathcal{G})$, output consensus is attained, i.e.,
$\lim_{t\to\infty} |y_i - y_j| = 0$, $\forall i, j$.

Isolated Equilibrium

- Example of non-isolated equilibriums:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = -\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix} = -L(\mathcal{P}_2) \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$



Isolated

The equilibrium $x_e \in \mathcal{A}$ is isolated for nonsingular A if $f(x)$ is differentiable at x_e and $\frac{d}{dx_i} f_i([x_e]_i) \neq 0$ for all $i = 1, \ldots, n$. The equilibrium is isolated for singular A if in addition

$$\sum_{i=1}^{n} \frac{v_i w_i}{\frac{d}{dx_i} f_i([x_e]_i)} \neq 0, \text{ for all } i = 1, \ldots, n.$$

Corollary

The equilibrium $x_e \in \mathcal{A}$ is isolated if $f_i(x)$ is differentiable at x_e and strictly increasing for all $i = 1, \ldots, n$.

Strictly Increasing Functions

Unique Equilibrium

If $f_i(x)$ is differentiable and strictly increasing for all $i = 1, \ldots, n$ then, for A nonsingular, $x_e = f^{-1}(0)$ is the unique stable equilibrium. For A singular, $x_e = f^{-1}(\beta v)$ is the unique equilibrium where $\beta \in \mathbb{R}$ "uniquely" satisfies $w^T x_e = w^T x_0$ such that $Av = 0$ and $w^T A = 0$.

Consensus

For $A = L(\mathcal{G})$, if $f_1(x) = f_2(x) = \cdots = f_n(x)$ is differentiable and strictly increasing then $\lim_{t\to\infty} |x_i - x_j| \to 0$, $\forall i, j$, i.e. internal state consensus is attained. Further if $\mathcal{G}$, is balanced then average internal state consensus is attained.

Extension

- Modified Dynamics with the addition of a nonlinear term,

$$\dot{x}_i(t) = -a_{ii} f_i(x_i(t)) + \sum_{j \neq i} a_{ij} f_j(x_j(t)) - g_i(x_i)$$

$$\dot{x} = -Ay - g(x) \text{ where } y_i = f_i(x_i),$$

  giving an equilibrium set $\mathcal{B} = \{x \in \mathbb{R}^n | Af(x) + g(x) = 0\}$
- Additional assumptions: $g_i(\cdot) \in \mathbb{R} \to \mathbb{R}$ is continuous and $f_i(x) g_i(x) \geq 0$

Stability Lemma

For the model, $x(t) \to \mathcal{B}$. If the intersection of the invariant set of the dynamics and $\mathcal{B}$ is composed of a finite number of isolated points then $x(t) \to x_e$ for some $x_e \in \mathcal{B}$.

- Application to neural networks: where $g_i(x_i) = \gamma_i x_i$, $\gamma_i > 0$, $f_i(\cdot)$ is increasing and $f_i(0) = 0$. Hence, $\mathcal{B} = \{\mathbf{0}\}$ and the origin is globally asymptotically stable

Conclusion

- Established sufficient conditions for stability of a large class of nonlinear dynamic networks
- Explored its equilibrium set
- Discussed the results' implications for output consensus
- Extended the model with the addition of an agent dependent nonlinear term
- Future work involves the introduction of control terms into the dynamics