On Strong Structural Controllability of Networked Systems: A Constrained Matching Approach

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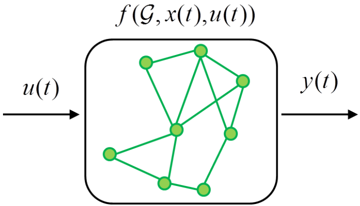
The Network in the Dynamics

General Dynamics

$$\dot{x}(t) = f(\mathcal{G}, x(t), u(t))$$
$$y(t) = g(\mathcal{G}, x(t), u(t))$$



Effective interfaces:

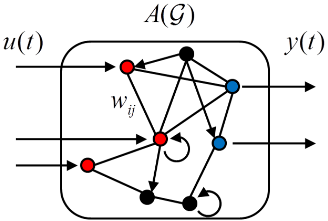
Network	System Dynamics
Effective resistance	$\mathcal{H}_2$ norm
Automorphisms	Homogeneity
Graph products/factorization	Controllability composition/decomposition
Bipartite Matching	Structural controllability

The Network in the Dynamics

- First Order, Linear Time Invariant model

$$\dot{x}_i(t) = \sum_{i \sim j} \pm w_{ij} x_j(t) + u_i(t)$$

$$y_i(t) = x_i(t)$$



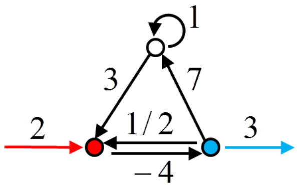
Dynamics

$$\dot{x}(t) = A(\mathcal{G})x(t) + B(S)u(t)$$
$$y(t) = C(R)x(t)$$

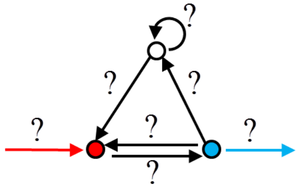
- $A(\mathcal{G})$: e.g. Z-matrix, Laplacian ($w_{ii} = -\sum w_{ij}$) and Advection matrices ($w_{ii} = -\sum w_{ji}$)
- Input node set $S = \{v_i, v_j, \dots\}$, $B(S) = [e_i, e_j, \dots]$
- Output node set $R = \{v_p, v_q, \dots\}$, $C(R) = [e_p, e_q, \dots]^T$

Flavors of Network Controllability

(1) General Controllability: Based on $A(\cdot)$ and $\mathcal{G}$

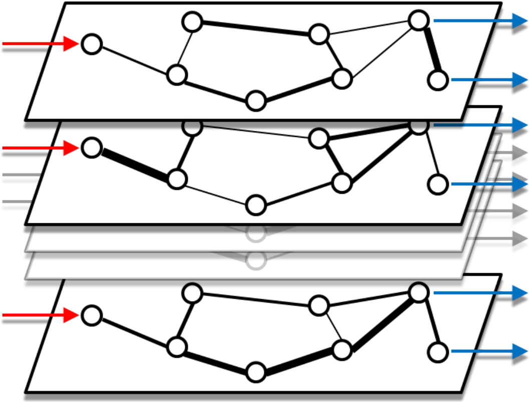


(2) Structural Controllability: Based on $\mathcal{G}$ alone



Structural Controllability

- A pair $(A(\mathcal{G}), B(S))$ is weak/strong structurally controllable (s-controllable), with weak/strong inputs S, if over every possible weighting of graph $\mathcal{G}$ it has one/all controllable realization(s)



Conceived: Lin '74, Mayeda and Yamada '79

Recently: Liu *et al.* '11, Reinschke *et al.* '92, Bowden *et al.* '12

Pattern Representation

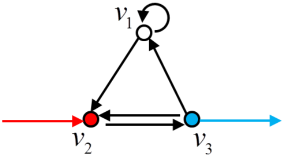
- A pattern matrix **A** is a matrix composed of zeros and crosses. A realization A of **A** maintains the zero structure
- $\mathbf{A}(\mathcal{G})$ defined s.t. one realization is the adjacency matrix $A(\mathcal{G})$ (Similarly for $\mathbf{B}(S)$ and $\mathbf{C}(R)$)

Patterned Dynamics

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

Here, $A \in \mathbf{A}(\mathcal{G})$, $B \in \mathbf{B}(S)$, $C \in \mathbf{C}(R)$.

Example:

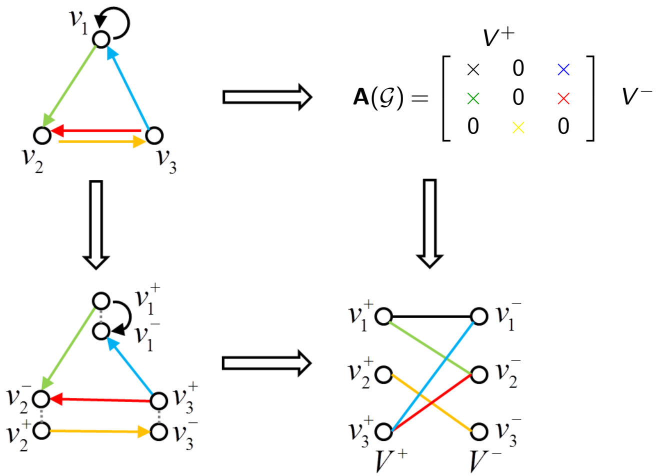


$$\mathbf{A}(\mathcal{G}) = \begin{bmatrix} \times & 0 & \times \\ \times & 0 & \times \\ 0 & \times & 0 \end{bmatrix}, \mathbf{B}(S) = \begin{bmatrix} 0 \\ \times \\ 0 \end{bmatrix}, \mathbf{C}(R) = \begin{bmatrix} 0 \\ 0 \\ \times \end{bmatrix}$$

- A pair $(\mathbf{A}(\mathcal{G}), \mathbf{B}(S))$ is weak/strong s-controllable, with weak/strong inputs S, if it has one/all controllable realization(s) (A, B)

Bipartite Representation

- Bipartite representation $\mathcal{H} = (V^+, V^-, E)$ of $\mathbf{A}(\mathcal{G}) \in \mathbb{R}^{p \times q}$

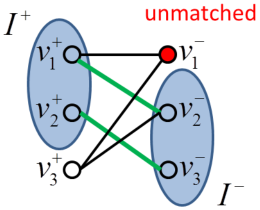


Rank and Matchings

- (A, B) is controllable iff $\text{rank}[A - \lambda I, B] = n$ for all eigenvalues λ of A

Combinatorial Criteria:		**Rank Criteria:**
$\mathbf{A}$ has a t-matching	$\Longrightarrow$	$\exists A \in \mathbf{A}$ with $\text{rank}(A) \geq t$

- $\mathbf{A}$ has a *t-matching* if there are t edges in $\mathcal{H} = (V^+, V^-, E)$ between $I^+ \subseteq V^+$ and $I^- \subseteq V^-$ ($|I^+| = |I^-| = t$) where no two edges share a node
- Nodes in $V^- \setminus I^-$ are called unmatched

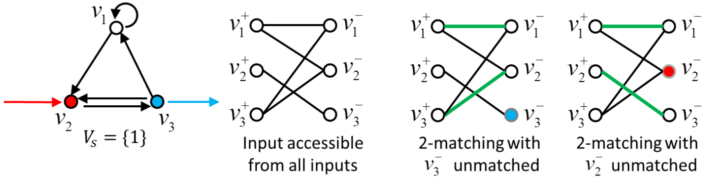


2-matching

Weak S-Controllability (Liu *et al.*)

Weak Inputs

S is weak iff $\mathbf{A}$ has an $(n - |S|)$-matching with S unmatched and input accessible.



$S = \{2\}$, $R = \{3\}$ are weak

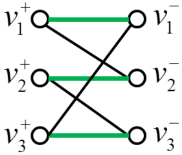
Weak Features

Weak Inputs

S is weak iff **A** has an $(n - |S|)$-matching with S unmatched and input accessible.

- Efficient algorithms for maximum bipartite matching
  - Deterministic $\mathcal{O}\left(\sqrt{|V|}|E|\right)$, Probabilistic $\mathcal{O}\left(|V|^{2.376}\right)$

- S is generically controllable but real-world systems can be atypical, e.g., undirected unweighted consensus
  - Adding edges tends to improve weak s-controllability



A self-damped network

Rank and Matchings Revisited

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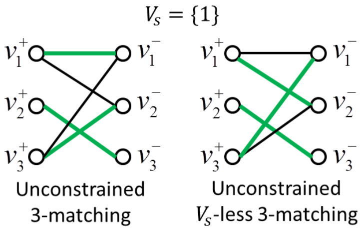
Combinatorial Criteria:		Rank Criteria:
$\mathbf{A}$ has a t-matching	$\Longrightarrow$	$\exists A \in \mathbf{A}$ with $\text{rank}(A) \geq t$
$\mathbf{A}$ has a constrained t-matching		$\forall A \in \mathbf{A}$, $\text{rank}(A) \geq t$

- A t-matching is constrained if it is the only t-matching between I^+ and I^-
- A matching is V_s-less if it contains no edges correponding to self loops, i.e., $\{v_i^+, v_i^-\}$.



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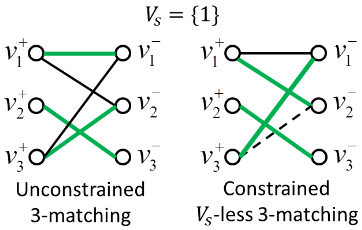
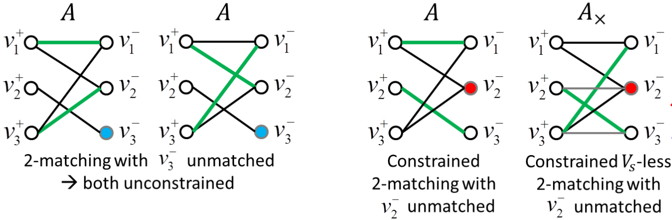


Strong inputs

S is strong iff $\mathbf{A}$ has a constrained $(n-|S|)$-matching with S unmatched and $\mathbf{A}_{\times}$ has a constrained V_s-less $(n-|S|)$-matching with S unmatched.

- Pattern matrix $\mathbf{A}_{\times}$ is formed by placing crosses along the diagonal of $\mathbf{A}$



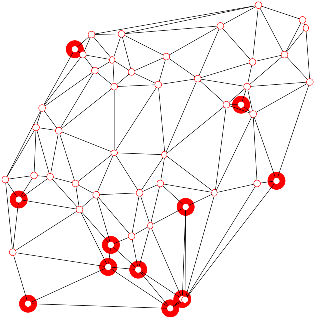
$S = \{2\}$ is strong
$R = \{3\}$ is not

Strong Features

- A strongly controllable input set S can be considered a type of *controllability robustness*
- For connected networks, adding edges tends to worsen strong s-controllability

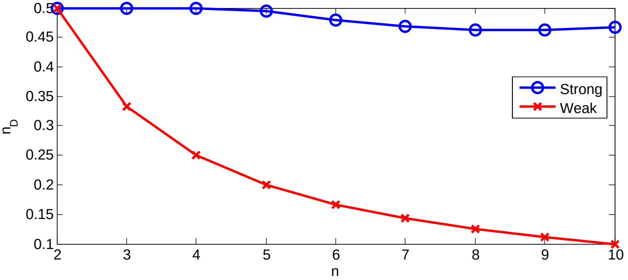
Algorithms:

- Golumbic (2001) - $\mathcal{O}(|V|+|E|)$ to check a matching is constrained
- Golumbic (2001) - NP-complete to find a maximum constrained matching
- Misha (2011) - Polynomial time algorithm to approximate a maximum constrained matching can do no better than $\frac{1}{2\sqrt[3]{9}}|V|^{\frac{1}{3}-\varepsilon}$ for any $\varepsilon > 0$
- We have an $\mathcal{O}(|V|^2)$ algorithm to check if S is strong and to find a (not-necessarily minimal) strong input set



Self-damped Undirected Networks

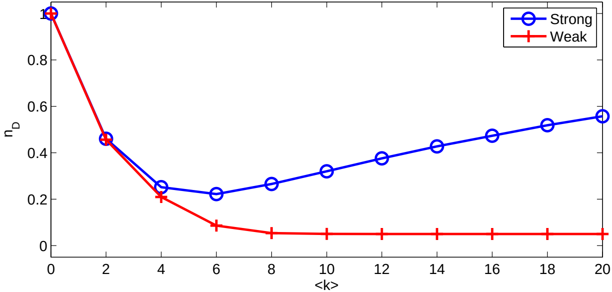
- Tested all undirected connected graphs for $n \leq 10$
- For S a minimum cardinality weak/strong set, $n_D := \frac{|S|}{|V|}$



- Smallest weak/strong S is a lower/upper bound on the smallest controllable input set for an arbitrary realization
- The only strong single input is the end nodes of a path graph
- The only realization requiring a strong $n-1$ input set is the complete graph

Directed Erdős-Rényi random networks

- Randomly generated on n nodes with a directed edge $(i,j) \in E$ existing with probability p and mean degree $\langle k \rangle = 2np$
- Tested 1200 graphs on 20 nodes for each $\langle k \rangle = 2, 4, \ldots, 20$



- $k_c = 2 \log n \approx 6$ is a sharp threshold for the disoriented connectedness of Erdős-Rényi random networks
- $k = k_c$ presents on average the minimum strong input set

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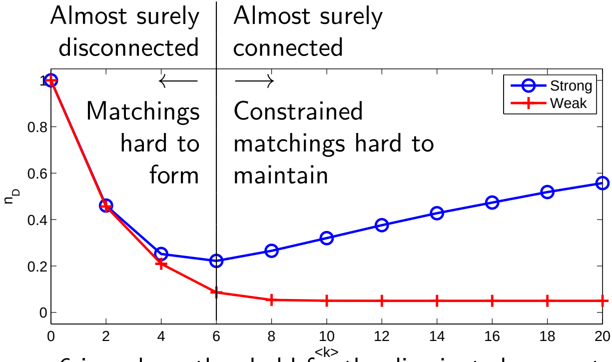

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Conclusion

- Linked strong s-controllability to a bipartite matching property and as such illustrated the computational challenges of the problem
- Compared features of weak and strong inputs
- Provided an efficient algorithm to generate strong inputs
- Brief examination of self-damped undirected and Erdős-Rényi networks
- Future Direction: Output weak and strong s-controllability, degree of controllability, edge s-controllability

Strong S-Controllability: Rough Proof

Rank test

(A, B) is controllable iff $[A - \lambda I, B]$ has full column rank for every eigenvalue λ of A.

- $[\mathbf{A}, \mathbf{B}]$ is full rank iff $\exists$ permutation matrices P_1 and P_2, s.t.,

$$P_1 [\mathbf{A}, \mathbf{B}] P_2 = \begin{bmatrix} \otimes & \cdots & \otimes & \times & 0 & \cdots & 0 \\ \vdots & & \vdots & \ddots & \times & \ddots & \vdots \\ & & & & \ddots & \ddots & 0 \\ \otimes & \cdots & \otimes & \cdots & \cdots & \otimes & \times \end{bmatrix},$$

where $\otimes$-elements can be either zero or crosses

For $\lambda = 0$

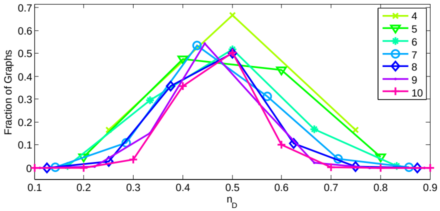
- $[\mathbf{A}, \mathbf{B}(S)]$ has this form iff $[\mathbf{A}, \mathbf{B}(S)]$ has a constrained n-matching
- Removing S rows of $[\mathbf{A}, \mathbf{B}(S)]$ leaves a constrained $(n - |S|)$-matching
- Implies $\mathbf{A}$ has a constrained $(n - |S|)$-matching with S unmatched
- Similarly for $\lambda = \times$, and $\mathbf{A}_\times$

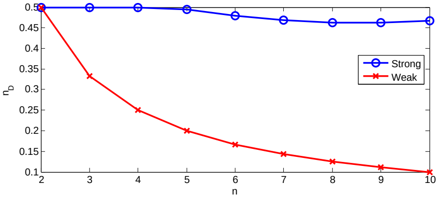
Self-damped Undirected Networks

Self-damped

Given a maximum constrained self-less matching of $\mathbf{A}_\times$ with unmatched nodes S. Then, S is strong with minimum cardinality.

- Tested all undirected connected graphs for $n \leq 10$
- For S a minimum cardinality strong set, $n_D := |S|/|V|$





- The only strong single input for a connected self-damped undirected network is the end nodes of a path graph
- The only connected self-damped undirected network requiring a strong $n-1$ input set is the complete graph

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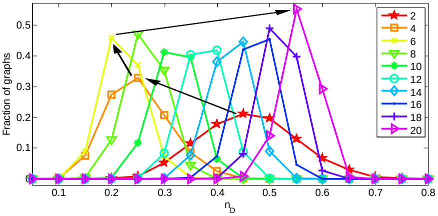

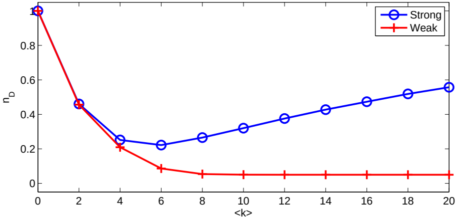

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