On Strong Structural Controllability of Networked Systems: A Constrained Matching Approach

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The Network in the Dynamics

### General Dynamics

\[
\dot{x}(t) = f(G, x(t), u(t)) \\
y(t) = g(G, x(t), u(t))
\]

### Effective interfaces:

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The Network in the Dynamics

- First Order, Linear Time Invariant model

\[
\dot{x}_i(t) = \sum_{i \sim j} \pm w_{ij} x_j(t) + u_i(t)
\]

\[
y_i(t) = x_i(t)
\]

Dynamics

\[
\dot{x}(t) = A(G)x(t) + B(S)u(t)
\]

\[
y(t) = C(R)x(t)
\]

- \(A(G)\): e.g. Z-matrix, Laplacian (\(w_{ii} = -\sum w_{ij}\)) and Advection matrices (\(w_{ii} = -\sum w_{ji}\))
- Input node set \(S = \{v_i, v_j, \ldots\}\), \(B(S) = [e_i, e_j, \ldots]\)
- Output node set \(R = \{v_p, v_q, \ldots\}\), \(C(R) = [e_p, e_q, \ldots]^T\)
(1) General Controllability: Based on $A(\cdot)$ and $G$

(2) Structural Controllability: Based on $G$ alone
Structural Controllability

- A pair \((A(G), B(S))\) is weak/strong structurally controllable (s-controllable), with weak/strong inputs \(S\), if over every possible weighting of graph \(G\) it has one/all controllable realization(s)

Conceived: Lin ’74, Mayeda and Yamada ’79
Recently: Liu et al. ’11, Reinschke et al. ’92, Bowden et al. ’12
A pattern matrix $A$ is a matrix composed of zeros and crosses. A realization $A$ of $A$ maintains the zero structure.

$A(G)$ defined s.t. one realization is the adjacency matrix $A(G)$ (Similarly for $B(S)$ and $C(R)$).

Example:

\[
A(G) = \begin{bmatrix}
\times & 0 & \times \\
\times & 0 & \times \\
0 & \times & 0
\end{bmatrix},
B(S) = \begin{bmatrix}
0 \\
\times \\
0
\end{bmatrix},
C(R) = \begin{bmatrix}
0 \\
0 \\
\times
\end{bmatrix}
\]

A pair $(A(G), B(S))$ is weak/strong s-controllable, with weak/strong inputs $S$, if it has one/all controllable realization(s) $(A, B)$.
Bipartite Representation

- Bipartite representation $\mathcal{H} = (V^+, V^-, E)$ of $A(G) \in \mathbb{R}^{p \times q}$

$$A(G) = \begin{bmatrix} V^+ \\ \times & 0 & \times \\ \times & 0 & \times \\ 0 & \times & 0 \end{bmatrix} V^-$$
(A, B) is controllable iff \( \text{rank} [A - \lambda I, B] = n \) for all eigenvalues \( \lambda \) of \( A \)

**Combinatorial Criteria:**

\( A \) has a \( t \)-matching \( \implies \) \( \exists A \in A \) with \( \text{rank}(A) \geq t \)

**Rank Criteria:**

\( A \) has a \( t \)-matching if there are \( t \) edges in \( \mathcal{H} = (V^+, V^-, E) \) between \( I^+ \subseteq V^+ \) and \( I^- \subseteq V^- \) \((|I^+| = |I^-| = t)\) where no two edges share a node

Nodes in \( V^- \setminus I^- \) are called unmatched

![Diagram](image)
Weak S-Controllability (Liu et al.)

Weak Inputs

$S$ is weak iff $A$ has an $(n - |S|)$-matching with $S$ unmatched and input accessible.

$S = \{2\}, \ R = \{3\}$ are weak
Weak Features

Weak Inputs

$S$ is weak iff $A$ has an $(n - |S|)$-matching with $S$ unmatched and input accessible.

- Efficient algorithms for maximum bipartite matching
  - Deterministic $O\left(\sqrt{|V||E|}\right)$, Probabilistic $O\left(|V|^{2.376}\right)$

- $S$ is generically controllable but real-world systems can be atypical, e.g., undirected unweighted consensus
  - Adding edges tends to improve weak s-controllability

A self-damped network

$\begin{align*}
  v_1^+ & \quad v_1^- \\
  v_2^+ & \quad v_2^- \\
  v_3^+ & \quad v_3^-
\end{align*}$
(A, B) is controllable iff \( \text{rank}[A - \lambda I, B] = n \) for all eigenvalues \( \lambda \) of \( A \)

**Combinatorial Criteria:**

- \( A \) has a \( t \)-matching
- \( A \) has a constrained \( t \)-matching

**Rank Criteria:**

\[
\exists A \in A \text{ with } \text{rank}(A) \geq t
\]
\[
\forall A \in A, \text{ rank}(A) \geq t
\]

- A \( t \)-matching is **constrained** if it is the only \( t \)-matching between \( I^+ \) and \( I^- \)
- A matching is **\( V_s \)-less** if it contains no edges corresponding to self loops, i.e., \( \{ v_i^+, v_i^- \} \).

Unconstrained 3-matching

Unconstrained \( V_s \)-less 3-matching

\( V_s = \{1\} \)
(A, B) is controllable iff $\text{rank}[A - \lambda I, B] = n$ for all eigenvalues $\lambda$ of $A$

**Combinatorial Criteria:**
- $A$ has a $t$-matching
- $A$ has a constrained $t$-matching

**Rank Criteria:**
- $\exists A \in A$ with $\text{rank}(A) \geq t$
- $\forall A \in A$, $\text{rank}(A) \geq t$

- A $t$-matching is **constrained** if it is the only $t$-matching between $I^+$ and $I^-$
- A matching is $V_s$-less if it contains no edges corresponding to self loops, i.e., $\{v_i^+, v_i^-, v_i^+, v_i^-\}$.
- $V_s = \{1\}$

Unconstrained 3-matching

Constrained $V_s$-less 3-matching
**Strong S-Controllability**

**Strong inputs**

\( S \) is strong iff \( A \) has a constrained \((n - |S|)\)-matching with \( S \) unmatched and \( A_x \) has a constrained \( V_s \)-less \((n - |S|)\)-matching with \( S \) unmatched.

- Pattern matrix \( A_x \) is formed by placing crosses along the diagonal of \( A \)

\[
\begin{align*}
A & \quad & A & \quad & A & \quad & A_x \\
\mathbf{v}_1^- & \quad & \mathbf{v}_1^- & \quad & \mathbf{v}_1^- & \quad & \mathbf{v}_1^- \\
\mathbf{v}_2^- & \quad & \mathbf{v}_2^- & \quad & \mathbf{v}_2^- & \quad & \mathbf{v}_2^- \\
\mathbf{v}_3^- & \quad & \mathbf{v}_3^- & \quad & \mathbf{v}_3^- & \quad & \mathbf{v}_3^- \\
\mathbf{v}_3^+ & \quad & \mathbf{v}_3^+ & \quad & \mathbf{v}_3^+ & \quad & \mathbf{v}_3^+ \\
\end{align*}
\]

\( S = \{2\} \) is strong

\( R = \{3\} \) is not
A strongly controllable input set $S$ can be considered a type of controllability robustness.

For connected networks, adding edges tends to worsen strong s-controllability.

**Algorithms:**

- Golumbic (2001) - $O(|V| + |E|)$ to check a matching is constrained.
- Golumbic (2001) - NP-complete to find a maximum constrained matching.
- Misha (2011) - Polynomial time algorithm to approximate a maximum constrained matching. Can do no better than $\frac{1}{2^{3\sqrt{9}}}|V|^{\frac{1}{3} - \varepsilon}$ for any $\varepsilon > 0$.
- We have an $O(|V|^2)$ algorithm to check if $S$ is strong and to find a (not-necessarily minimal) strong input set.
Self-damped Undirected Networks

- Tested all undirected connected graphs for $n \leq 10$
- For $S$ a minimum cardinality weak/strong set, $n_D := \frac{|S|}{|V|}$

![Graph showing $n_D$ vs $n$]

- Smallest weak/strong $S$ is a lower/upper bound on the smallest controllable input set for an arbitrary realization
- The only strong single input is the end nodes of a path graph
- The only realization requiring a strong $n - 1$ input set is the complete graph
Directed Erdős-Rényi random networks

- Randomly generated on $n$ nodes with a directed edge $(i,j) \in E$ existing with probability $p$ and mean degree $\langle k \rangle = 2np$
- Tested 1200 graphs on 20 nodes for each $\langle k \rangle = 2, 4, \ldots, 20$

$k_c = 2 \log n \approx 6$ is a sharp threshold for the disoriented connectedness of Erdős-Rényi random networks

$k = k_c$ presents on average the minimum strong input set
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\[ k = k_c \text{ presents on average the minimum strong input set} \]
Conclusion

- Linked strong s-controllability to a bipartite matching property and as such illustrated the computational challenges of the problem.
- Compared features of weak and strong inputs.
- Provided an efficient algorithm to generate strong inputs.
- Future Direction: Output weak and strong s-controllability, degree of controllability, edge s-controllability.
Strong S-Controllability: Rough Proof

**Rank test**

\((A, B)\) is controllable iff \([A - \lambda I, B]\) has full column rank for every eigenvalue \(\lambda\) of \(A\).

- \([A, B]\) is full rank iff \(\exists\) permutation matrices \(P_1\) and \(P_2\), s.t.,

\[
P_1 [A, B] P_2 = \begin{bmatrix}
\otimes & \cdots & \otimes & \times & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \times & \ddots & \vdots & \vdots & \vdots \\
\otimes & \cdots & \otimes & \cdots & \cdots & \otimes & \times
\end{bmatrix},
\]

where \(\otimes\)-elements can be either zero or crosses

For \(\lambda = 0\)

- \([A, B(S)]\) has this form iff \([A, B(S)]\) has a constrained \(n\)-matching
- Removing \(S\) rows of \([A, B(S)]\) leaves a constrained \((n - |S|)\)-matching
- Implies \(A\) has a constrained \((n - |S|)\)-matching with \(S\) unmatched
- Similarly for \(\lambda = \times\), and \(A_{\times}\)
Self-damped Undirected Networks

Self-damped

Given a maximum constrained self-less matching of $A_x$ with unmatched nodes $S$. Then, $S$ is strong with minimum cardinality.

- Tested all undirected connected graphs for $n \leq 10$
- For $S$ a minimum cardinality strong set, $n_D := |S|/|V|$

The only strong single input for a connected self-damped undirected network is the end nodes of a path graph.

The only connected self-damped undirected network requiring a strong $n-1$ input set is the complete graph.
Directed Erdős-Rényi random networks

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