Semi-Autonomous Networks: Effective Control of Networked Systems through Design, Modeling, and Protocols

Airlie Chapman

Distributed Space Systems Lab (DSSL)
Robotics, Aerospace and Information Networks (RAIN)

University of Washington

Advisor: Mehran Mesbahi
Semi-Autonomous Networked Systems

Arlie Chapman (Distributed Space Systems Lab (DSSL) Robotics, Aerospace and Information Networks (RAIN))
Effective control...

- Network measures?
- Favorable topological features?
- Local features?

Network scalability?
Network decomposition?
Invariant system properties?

Extensions to agreement?
Extensions to nonlinear?
Families of network protocols?

Approach

- Design
- Modeling
- Protocols

Semi-Autonomous Networks
Approach

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Semi-Autonomous Networks

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Semi-Autonomous Networks

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Design

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Modeling

Extensions to agreement?
Extensions to nonlinear?
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Protocols
Effective control...
The Network in the Dynamics

- Node $i$ dynamics

$$
\dot{x}_i(t) = -w_{ii}x_i(t) + \sum_{i\sim j} w_{ij}x_j(t) + u_i(t)
$$

$$
y_i(t) = x_i(t)
$$

Dynamics

$$
\dot{x}(t) = A(G)x(t) + B(S)u(t)
$$

$$
y(t) = C(R)x(t)
$$

- $A(G)$: Encodes the graph structure, e.g. Consensus: undirected unweighted simple $G$ and $A(G) = A(G) - \text{diag}(A(G)1)$
- Input node set $S = \{v_i, v_j, \ldots\}$, $B(S) = [e_i, e_j, \ldots]$
- Output node set $R = \{v_p, v_q, \ldots\}$, $C(R) = [e_p, e_q, \ldots]^T$
The $H_2$ norm $\|G(s)\|_2^2$ for Leader-Follower Consensus

Energy at the output from the origin due to a unit impulse $u(t)$, $\int_0^\infty y(t)^Ty(t)$

$$\|G(s)\|_2^2 = \frac{1}{2} \sum_{v_i \in \mathcal{N}(S)} E_{\text{eff}}(v_i)$$

where $v_i \in \mathcal{N}(S)$ if it neighbors a leader

**Effective Resistance:** $E_{\text{eff}}(v_i)$ is the voltage drop between $v_i$ and $s^*$, when a 1 Amp current source is connected across them.

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Design changes through Rewiring

- Circuit theory provides a way to predictably alter structure

Example: Wind Gust Alleviation

Beyond Linear Consensus

General Dynamics

\[ \dot{x}(t) = f(G, x(t), u(t)) \]
\[ y(t) = g(G, x(t), u(t)) \]

Example:
Human-Swarm Interaction

**Cartesian Product Networks**

**Networks within Networks**

\[ G_1 \square G_2 \]

**Approximate Product Networks**

\[ G_1 \square G_2 \]

- Invariant features over the factor networks:
  - **Controllability**, stability, trajectory subspaces


Road Ahead - Strong Structural Controllability

What can one say about controllability based on the $G$ alone?

... Structural Controllability


Factoring the Dynamics via Cartesian Products
Graph Products: Networks within Networks

- Many ways to compose graphs $G$ and $H$
  - Cartesian product $G□H$
  - Tensor product $G × H$
  - Strong product $G △ H$
  - Lexicographic product $G • H$
  - Rooted product $G ◦ H$
  - Corona product $G ⊙ H$
  - Star product $G ⋆ H$

- How does modularity of the network manifest itself as modularity within the state dynamics?

**Cartesian Product:** $(\text{Graphs, } □) \rightarrow (\text{Dynamics, } ⊗)$
Graph Product Examples

- **Periodic Structures:**
  e.g., hypercube multiprocessors, building trusses

- **Compartmental Networks:**
  e.g., air traffic networks, chemical reactions

- **Constant degree expander graphs:**
  e.g., computer networks, sorting networks, cryptography

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**Australian Academic Research Network (AARNET)**

*Cite: Parsonage et al. “Generalized Graph Products for Network Design and Analysis”, 2011.*
Graph Cartesian Product

- Cartesian product $G \square H$
- Vertex set: $V(G \square H) = V(G) \times V(H)$
- Edge set: $(x_1, x_2) \sim (y_1, y_2)$ is in $G \square H$
  - if $x_1 \sim y_1$ and $x_2 = y_2$ or $x_1 = y_1$ and $x_2 \sim y_2$
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Graph Factorization

- A graph can be factored as well as composed...

Theorem (Sabidussi 1960)

Every connected graph can be factored as a Cartesian product of prime graphs. Moreover, such a factorization is unique up to reordering of the factors.

- Primes: $G = G_1 \square G_2$ implies that either $G_1$ or $G_2$ is $K_1$
  - Number of prime factors is at most $\log |G|

- Algorithms
  - Feigenbaum (1985) - $\mathcal{O}(|V|^{4.5})$
  - Winkler (1987) - $\mathcal{O}(|V|^4)$ from isometrically embedding graphs by Graham and Winkler (1985)
  - Feder (1992) - $\mathcal{O}(|V||E|)$
  - Imrich and Peterin (2007) - $\mathcal{O}(|E|)$
  - C++ implementation by Hellmuth and Staude
Input and Output Set Product

\[ S_1 \times R_1 \times S_2 \times R_2 \]
Controllability

- Dynamics are **controllable** if for any $x(0)$, $x_f$ and $t_f$ there exists an input $u(t)$ such that $x(t_f) = x_f$.


- Challenging to establish for large networks

- Known families of controllable graphs for selected inputs
  - Paths (Rahmani and Mesbahi ’07)
  - Circulants (Nabi-Abdolyousefi and Mesbahi ’12)
  - Grids (Parlengeli and Notarsefano ’11)
  - Distance regular graphs (Zhang *et al.* ’15)
The dynamics

\[ \dot{x}(t) = -A(\prod G_i)x(t) + B(\prod S_i)u(t) \]
\[ y(t) = C(\prod R_i)x(t) \]

where \( A(\prod G_i) \) has simple eigenvalues is controllable/observable if and only if

\[ \dot{x}_i(t) = -A(G_i)x_i(t) + B(S_i)u_i(t) \]
\[ y_i(t) = C(R_i)x_i(t) \]

is controllable/observable for all \( i \).
Controllability Factorization - Idea of the Proof

Popov-Belevitch-Hautus (PBH) test

$(A, B)$ is uncontrollable if and only if there exists a left eigenvalue-eigenvector pair $(\lambda, v)$ of $A$ such that $v^T B = 0$.

**Eigenvalue and eigenvector relationship:**

<table>
<thead>
<tr>
<th>Eigenvalue</th>
<th>$\lambda_i$</th>
<th>$\mu_j$</th>
<th>$\lambda_i + \mu_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eigenvector</td>
<td>$v_i$</td>
<td>$u_j$</td>
<td>$v_i \otimes u_j$</td>
</tr>
</tbody>
</table>

Also $(v_i \otimes u_i)^T (B(S_1) \otimes B(S_2)) = v_i^T B(S_1) \otimes u_i^T B(S_2)$

The proof follows from these observations.
Theorem 2: Layered Controllability

The dynamics

$$\dot{x}(t) = -A(\prod G_i)x(t) + B(\prod S_i)u(t)$$

$$y(t) = C(\prod R_i)x(t)$$

where $A(G_i)$ is diagonalizable for all $i$ and $S_i = R_i = V(G_i)$ for $i = 2, \ldots, n$ is controllable/observable if and only if

$$\dot{x}_1(t) = -A(G_1)x_1(t) + B(S_1)u_1(t)$$

$$y_1(t) = C(R_1)x_1(t)$$

is controllable/observable.
Example

Intra-Family Network

Product Control: Father 14
Layered Control: All fathers or all family 14

Inter-Family Network

Full Network
Uncontrollability through Symmetry

Proposition (Rahmani and Mesbahi 2006)

\((A(G), B(S))\) is uncontrollable if there exists an automorphism of \(G\) which fixes all inputs in the set \(S\) (i.e., \(S\) is not a determining set.)

The determining number of a graph \(G\), denoted \(Det(G)\), is the smallest integer \(r\) so that \(G\) has a determining set \(S\) of size \(r\).

Corollary

\((A(G), B(S))\) is uncontrollable if \(|S| < Det(G)\).
Breaking Symmetry

Automorphism group for graph Cartesian products

The automorphisms for a connected $\mathcal{G}$ is generated by the automorphisms of its prime factors.

Proposition: Automorphism group for graph Cartesian products

For controllable pairs $(A(\mathcal{G}_1), B(S_1))$ and $(A(\mathcal{G}_2), B(S_2))$ where $|S_1| = \text{Det}(\mathcal{G}_1)$ and $|S_2| = 1$. Then $S = S_1 \times S_2$ is the smallest input set such that $A(\mathcal{G}_1 \square \mathcal{G}_2, B(S))$ is controllable.
Strong Structural Controllability
As mentioned... it’s challenging to establish for large networks.

Two Approach:

1. General Controllability: Based on $A(\cdot)$ and $G$

2. Structural Controllability: Based on $G$ alone
A pair \((A(G), B(S))\) is \textit{weak/strong} structurally controllable (s-controllable), with \textit{weak/strong} inputs \(S\), if over every possible weighting of graph \(G\) it has \textit{one/all} controllable realization(s)

Conceived: Lin '74, Mayeda and Yamada '79
Recently: Liu et al. '11, Reinschke et al. '92
Structural Controllability

- A pattern matrix $A$ is a matrix composed of zeros and crosses. A realization $A$ of $A$ maintains the zero structure.
- $A(G)$ defined s.t. the adjacency of $G$, $A(G)$ is a realization (Similarly for $B(S)$ and $C(R)$).

Example:

A pair $(A(G), B(S))$ is weak/strong s-controllable, with weak/strong inputs $S$, if it has one/all controllable realization(s) $(A, B)$.
Bipartite Representation

- Bipartite representation $\mathcal{H} = (V^+, V^-, E)$ of $A(G) \in \mathbb{R}^{p \times q}$

\[
V^+ = \begin{bmatrix} 0 & \times & \times \\ \times & 0 & \times \\ 0 & \times & 0 \end{bmatrix}
\]

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The controllability criteria on the rank of $[A - \lambda I, B]$ is the main tool for establishing s-controllability.

A rank criteria can be linked to a combinatorial matching criteria.

Let $A$ be an $m \times n$ pattern matrix, and $t$ an integer, $1 \leq t \leq \min\{m, n\}$.

$t$-matchings and Rank

If $A$ has a $t$-matching, then there exists a matrix $A \in A$ with $\text{rank}(A) \geq t$.

- Required for Weak s-controllability
Bipartite Matching

- A has a \textit{t-matching} if there are \( t \) edges in \( \mathcal{H} = (V^+, V^-, E) \) between \( I^+ \subseteq V^+ \) and \( I^- \subseteq V^- (|I^+| = |I^-| = t) \) where no two edges share a node.
- Nodes in \( V^- - I^- \) are called **unmatched**.

![Diagram](image)
Weak S-Controllability (Liu et al.)

**Weak Inputs**

**Check:** $S$ is weak iff $A$ has an $(n - |S|)$-matching with $S$ unmatched and $S$ is input accessible.

**Find:** Given a maximum matching on $A$ with unmatched nodes $S_1$ and input accessible nodes $S_2$. For $S_1 \neq \emptyset$, $S_1 \cup S_2$ is weak, otherwise $S_2$ is weak.

- Efficient algorithms for maximum bipartite matching
  - Deterministic $O\left(\sqrt{|V||E|}\right)$, Probabilistic $O\left(|V|^{2.376}\right)$
- Smallest weak $S$ is a lower bound on smallest controllable input set
Let $A$ be an $m \times n$ pattern matrix, and $t$ an integer, $1 \leq t \leq \min\{m, n\}$

**$t$-matchings and Weak Rank**

If $A$ has a $t$-matching, then there exists a matrix $A \in A$ with $\text{rank}(A) \geq t$.

$\implies$ Weak $s$-controllability

**$t$-matchings and Strong Rank**

If $A$ has a constrained $t$-matching, then every matrix $A \in A$ has $\text{rank}(A) \geq t$.

$\implies$ Strong $s$-controllability
**Bipartite Matching**

- A has a \( t \)-matching if there are \( t \) edges in \( \mathcal{H} = (V^+, V^-, E) \) between \( I^+ \subseteq V^+ \) and \( I^- \subseteq V^- \) (\(|I^+| = |I^-| = t\)) where no two edges share a node.
- Nodes in \( V^- - I^- \) are called unmatched.

- A \( t \)-matching is **constrained** if it is the only \( t \)-matching between \( I^+ \) and \( I^- \).
- A matching is **Z-less** if it contains no edges of the form \( \{v_i^+, v_i^-\} \) where \( i \in Z \). If \( Z = V \) then the matching is **self-less**.
Strong S-Controllability

- Pattern matrix $A_\times$ is formed by placing crosses along the diagonal of $A$
- $V_s \subseteq V$ is the set of nodes with self-loops

**Strong inputs**

**Check:** $S$ is strong iff $A$ has a constrained $(n - |S|)$-matching and $A_\times$ has a constrained $V_s$-less $(n - |S|)$-matching, both with $S$ unmatched.

**Find:** Given a constrained matching on $A$ with unmatched $S_1$ and a constrained $V_s$-less matching on $A_\times$ with unmatched $S_2$. Then, $S_1 \cup S_2$ is strong.

- Smallest strong $S$ is an upper bound on smallest controllable input set
Golumbic (2001) - $\mathcal{O}(|V| + |E|)$ to check a matching is constrained

Golumbic (2001) - NP-complete to find a maximum constrained matching

Misha (2011) - Polynomial time algorithm to approximate a maximum constrained matching can do no better than $\frac{1}{2^{\frac{3}{\sqrt{9}}}}|V|^{\frac{1}{3} - \varepsilon}$ for any $\varepsilon > 0$

We have an $\mathcal{O}(\sqrt{|V|})$ algorithm to check if $S$ is strong and to find a (not-necessarily minimal) strong input set
Moving forward... S-Controllability

- Explore s-controllability for typical networked systems topologies
- Use s-controllability to identify vulnerable nodes and critical edges
- Link s-controllability to topological features
- Examining the “degree” of controllability
- Investigate output weak and strong s-controllability

Moving forward... Human-Swarm Interactions

Design distributed protocols to interpret coarse inputs from a human operator
Explore online algorithms to distributively optimize a global cost, e.g., dampen changing disturbances into the network by interconnection reweighting.

Conclusion

- Cartesian Product
  - Presented a factorization of controllability - a product and layered approach
  - Linked the factors symmetry to smallest controllable input set

- Strong Structural Controllability
  - Introduced constrained bipartite matching to establish strong inputs

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<th>Analysis Tool</th>
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Thanks to my committee and collaborators...

- Professor Mesbahi
- Professor Burke
- Professor Klavins
- Professor Morgansen
- Professor Thomas
- Professor Shea-Brown
- Dr Marzieh Nabi-Abdolyousefi
- Eric Schoof
- Ran Dai
- Saghar Hosseini