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Kronecker Product of Networked Systems and their Approximates

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***Abstract*—The paper presents a system theoretic analysis of network-of-networks which are formed from smaller factor networks via graph Kronecker products. We provide a compositional framework for extending trajectories, stabilizability, controllability and observability of the factor networks to that of the composite network-of-networks. We proceed to delve into the effectiveness of applying these composite features to approximate Kronecker product networks. Examples are provided throughout to illustrate the results.**

***Index Terms*—Approximate graph products; Network controllability; Network observability; Graph Kronecker product; Composite networks; Coordination algorithms**

I. Introduction

One of the central themes in control theory pertains to the analysis of properties that are preserved under composition of systems, mainly in terms of their cascade and feedback interconnections. Composition of linear systems is a common tool to form and analyze dynamic systems. For instance, control system design often involves judicious interconnection of atomic subsystems, including the plant, filters, sensors, actuators, and static nonlinearities, in order to provide favorable properties of the composed dynamic system.

A prime example of this is the use of passivity as a system property, preserved under feedback and parallel interconnections, with a multitude of ramifications for system synthesis [1]. Other examples include forming stable interconnected systems using stable atomic subsystems via the small-gain theorem, composite Lyapunov functions, and in the framework of compartmental systems [2]. Special forms of subsystem composition, such as series-parallel and feedback, has also been used to establishing controllability and observability in terms of controllability and observability of their subsystems [3].

Decomposition has been yet another facet of system analysis using a compositional perspective. Decomposition techniques provide compact descriptions of systems, providing both analytical and computational benefits. The minimal state-space realization method finds a state-space description of minimal dimension that describes a system [4]. Also, Jordan decomposition reveals the finite zero structure as well as the invariance properties of linear systems [5]. Other examples of the decomposition perspective include the Kalman decomposition and structural decomposition methods [6], [7].

The focus of the present paper is on a network-centric compositional theory for networked dynamic systems. Networked dynamic systems are an integral part of the technological world including power grids and information networks, as well as in areas such as biological and social systems. Much research in the area of networked systems has eventuated [8], [9], [10]. In order to provide a concrete utility of a compositional theory of networked systems, in this paper we focus on arguably the most basic properties of controlled processes, namely, their unforced dynamics, stabilizability, controllability and observability.

Controllability and observability of networked dynamic systems adopting consensus-type coordination algorithms have recently attracted the attention of researchers in a multitude disciplines [11], [12], [13], [14]. Network controllability becomes important when a networked system is influenced or observed by an external entity; such scenarios include networked robotic systems [15], [16], human-swarm interaction [17], network security [18], [19], and quantum networks [20].

Our results are applicable to control problems with underlying structures which form Kronecker products or approximate Kronecker products [21]. These types of systems are induced by, or eventuate through, many processes. An attraction of the Kronecker product is it exhibits a *network-of-networks* style feature where each node of one factor can be considered to contain a copy of another factor. It is this "fractal" nature of the product that has spurred its application in system sciences and biology [22]. We have recently explored other results in the area of network-of-networks using the Cartesian product [23], [24], specifically developing controllability results.

The organization of the paper is as follows. We begin by introducing relevant background material pertaining to notation, graphs, Kronecker graph products and Kronecker products. This is followed by the introduction of Kronecker-based network dynamics. We then proceed to examine these dynamics in the context of their unforced trajectory, stabilizability, controllability and observability. Examples are provided along with each result.

II. Background

We provide a brief background on constructs and models that will be used in this paper.

For column vector $v \in \mathbb{R}^p$, v_i or $[v]_i$ denotes the ith element. For matrix $M \in \mathbb{R}^{p\times q}$, $[M]_{ij}$ denotes the element in its ith row and jth column. We write $M \succ 0$ ($M \succeq 0$) if M is positive definite (semidefinite) matrix. A matrix M is nonnegative (positive), denoted $M \geq 0$ ($M > 0$) if all entries of M are nonnegative (positive). Further, $M \geq N$ ($M > N$) is equivalent to $M - N \geq 0$ ($M - N > 0$). For

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matrices $M \in \mathbb{R}^{n\times p}$ and $V \in \mathbb{R}^{n\times q}$, $M \perp V$ imply that $v^T M = 0$ for some nonzero v in the span of V, similarly, $M \not\perp [v_1, v_2, \ldots, v_n]$ implies that no such v exists. The notation $\|M\|$ denotes the 2-norm of matrix M and $\rho(M)$ denotes its maximum singular value.

A. Graphs

A weighted digraph $\mathcal{G} = (V, E, W)$ is characterized by a node set V with cardinality n, an edge set E comprised of ordered pairs of nodes with cardinality m, and a weight set W with cardinality m, where information flows from node i to j (node i is adjacent to node j) if $(i, j) \in E$ with edge weight $w_{ji} \in W$. The adjacency matrix is an $n \times n$ matrix with $[\mathcal{A}(\mathcal{G})]_{ij} = w_{ij}$ when $(j, i) \in E$ and $[\mathcal{A}(\mathcal{G})]_{ij} = 0$ otherwise.

B. Kronecker Graph Product (Direct Product)

There are a number of ways to synthesize large-scale networks from a set of smaller size graphs [25]. The Kronecker graph product (or Kronecker product for brevity) also known as the direct graph product is one such method and is defined for a pair of *factor* graphs $\mathcal{G}_1 = (V_1, E_1, W_1)$ and $\mathcal{G}_2 = (V_2, E_2, W_2)$ and denoted by $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$. The *product* graph $\mathcal{G}$ has the vertex set $V_1 \times V_2$ and there is an edge from vertex (i, p) to (j, q) in $V_1 \times V_2$ if and only if (p, q) is an edge of E_2 and (u, v) is an edge of E_1. The corresponding weight of an edge, if it exists, is $w_{((i,p),(j,q))} = w_{ij}w_{pq}$. The Kronecker product is commutative and associative, i.e., the products $\mathcal{G}_1 \times \mathcal{G}_2$ and $\mathcal{G}_2 \times \mathcal{G}_1$ are isomorphic; similarly $(\mathcal{G}_1 \times \mathcal{G}_2) \times \mathcal{G}_3$ and $\mathcal{G}_1 \times (\mathcal{G}_2 \times \mathcal{G}_3)$ are isomorphic.

An example of a Kronecker product of two factor graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ is displayed in Figure 1.

A graph is called *prime* if it cannot be decomposed into the product of non-trivial graphs, otherwise a graph is referred to as composite. Sabidussi [26] and Vizing [27] highlighted the fundamental nature of the primes noting that connected graphs decompose, although not uniquely, into primes, up to reordering. Further, Imrich [25] demonstrated that a digraph can be factored into primes in polynomial-time.

Hellmuth *et al.* in [28] presented an algorithm to recognize original factors of approximated graph products. The problem of approximating graph products is also of importance in a wide range of applications in biology [28].

Many features of the factors of a composite graph transfer to the composite graph, such as if one of the factors $\mathcal{G}_1$ and $\mathcal{G}_2$ are disconnected then so too is $\mathcal{G}_1 \times \mathcal{G}_2$. In this paper we show that when the composite graph underlies a dynamic system, many useful features of dynamics can be revealed by examining dynamics systems over the factor graphs.

C. Kronecker Product

Kronecker algebra (distinct from the Kronecker graph product) provides the main machinery for many of the results of this paper. Let matrix $A = [a_{ij}] \in \mathbb{R}^{m\times n}$ and $B = [b_{ij}] \in \mathbb{R}^{p\times q}$. The Kronecker product of A and B, denoted by $A \otimes B$, is essentially obtained by replacing the

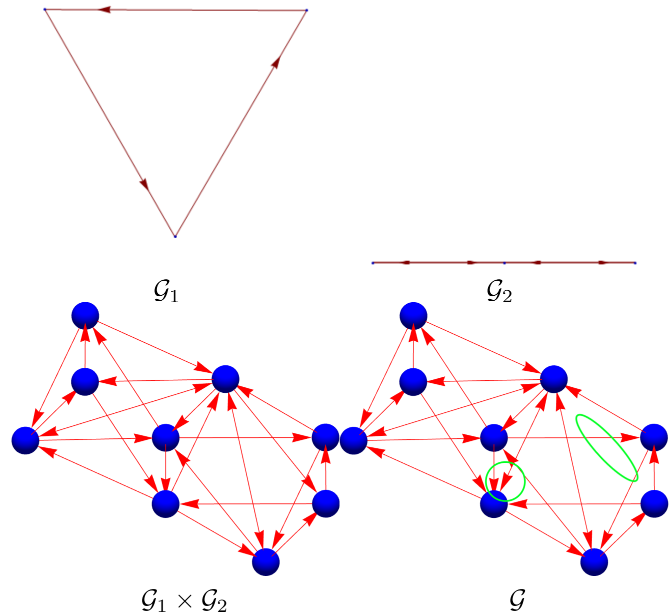

Figure 1. Factor graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ and composite Kronecker product graph $\mathcal{G}_1 \times \mathcal{G}_2$. Self loops are not drawn for clarity but appear on one node in $\mathcal{G}_1$ and all nodes in $\mathcal{G}_2$ ($\mathcal{G}_1 \times \mathcal{G}_2$ self loops can be divulged through the product). Graph $\mathcal{G}$ is approximately the graph product $\mathcal{G}_1 \times \mathcal{G}_2$ with edge differences highlighted.

ij-th entry of A by the matrix $a_{ij}B$, for every $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Hence

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \in \mathbb{R}^{mp\times nq}.$$

Kronecker products of two matrices have a number of useful properties, including

$$(A \otimes B)(C \otimes D) = AC \otimes BD$$

where $A \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{l\times k}$, $C \in \mathbb{R}^{m\times q}$, and $D \in \mathbb{R}^{k\times p}$. However, one of the remarkable properties of Kronecker products relates to the eigenvalue-eigenvector correspondence between the matrices A and B, on one hand, and $A \otimes B$, on the other. In particular, if u and v are right eigenvectors of A and B, respectively, with corresponding eigenvalues λ and μ, then

$$\begin{aligned}(A \otimes B)(u \otimes v) &= Au \otimes Bv \\ &= \lambda\mu(u \otimes v).\end{aligned}$$

Therefore $u \otimes v$ is a right eigenvector of $A \otimes B$ with corresponding eigenvalue $\lambda\mu$. Consequently, $\rho(A \otimes B) = \rho(A)\rho(B)$.

Another property of the Kronecker product is that given the eigenvalue/right-eigenvector (left-eigenvector) pairs of A and B as (λ_i, u_i) for $i = 1, \ldots, n$ and (μ_j, v_j) for $j = 1, \ldots, m$, respectively, then $(\lambda_i\mu_j, u_i \otimes v_j)$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ are the eigenvalue/eigenvector pairs of $A \otimes B$.

A feature that will be exploited in this paper is that the adjacency matrix of a Kronecker graph product can be represented in terms of its factor's adjacency matrices, specifically

$$\mathcal{A}(\mathcal{G}_1 \times \mathcal{G}_2) = \mathcal{A}(\mathcal{G}_1) \otimes \mathcal{A}(\mathcal{G}_2).$$

This follows directly from the definition of the Kronecker graph product.

We now proceed to introduce the Kronecker product dynamics. We then apply the presented background tools to analyze the new dynamics formed from applying Kronecker products to these graphs.

III. Model

There are a number of ways to construct the (system) matrix $A(\mathcal{G}) \in \mathbb{R}^{n\times n}$ associated with an n node graph $\mathcal{G}$. One option is the adjacency matrix $\mathcal{A}(\mathcal{G})$ which we have already touched upon in §II-A.

In this paper we consider networks with system matrices formed from matrices sympathetic to the Kronecker product. Specifically system matrices such that

$$A(\mathcal{G}_1 \times \mathcal{G}_2) = A(\mathcal{G}_1) \otimes A(\mathcal{G}_2),$$

denoted by $A(\cdot) \in \boldsymbol{A}_\otimes$. Two notable members of $\boldsymbol{A}_\otimes$ are the adjacency matrix itself and the row stochastic representation $A_s(\mathcal{G})$ of the adjacency defined for all i, j as

$$[A_s(\mathcal{G})]_{ij} = \frac{[\mathcal{A}(\mathcal{G})]_{ij}}{\sum_j [\mathcal{A}(\mathcal{G})]_{ij}}.$$

We will be exploring a compositional theory of factor systems

$$x_i(k+1) = A(\mathcal{G}_i)x_i(k) + B_i u_i(k), \quad y_i(k) = C_i x_i(k)$$

for $i = 1, \ldots, s$, where $A(\cdot) \in \boldsymbol{A}_\otimes$, which are composed together to form the composite system

$$\begin{aligned} x(k+1) &= A(\prod_\times \mathcal{G}_i)x(k) + \prod_\otimes B_i u(k) \\ y(k) &= \prod_\otimes C_i x_i(k). \end{aligned} \tag{1}$$

We often refer to such dynamics by specifying the triplet $(A(\mathcal{G}), B, C)$, or if only the inputs and outputs are of interest, by matrix pairs $(A(\mathcal{G}), B)$ and $(A(\mathcal{G}), C)$, respectively.

The Kronecker dynamics (1) appear naturally in many real world systems that exhibit 'fractal' like structure. These systems are also termed *hierarchically organized networks* [22] where a seed graph $\mathcal{G}_1$ recursively creates a community structures $\mathcal{G}_2$ at each node. In turn, the nodes of $\mathcal{G}_2$ become the seeds for graph $\mathcal{G}_3$ and so on. The associated inputs and outputs B_i and C_i at each stage, similarly act as seeds for future inputs and outputs.

Van Loan and Pitsianis [29] explored the Kronecker approximation of matrices that are not pure Kronecker products. He provided a computationally efficient method to find the optimal A_1 and A_2 such that $\|A - A_1 \otimes A_2\|$ is minimized under the 2-norm and Frobenius norm. In this paper we explore systems of the form

$$\begin{aligned} x(k+1) &= \left(A(\prod_\times \mathcal{G}_i) + \Delta\right) x(k) + \prod_\otimes B_i u(k) \\ y(k) &= \prod_\otimes C_i x_i(k), \end{aligned} \tag{2}$$

where $A(\cdot) \in \boldsymbol{A}_\otimes$, and approximate them using dynamics (1). For the current paper we assume that input and output matrices are in Kronecker form. We leave the examination of Kronecker approximations of the input and output matrices as future work.

The following gives an example Kronecker approximation.

Example 1. Consider the composite graph $\mathcal{G} = (V, E)$ with $V(\mathcal{G}) = V(\mathcal{G}_1 \times \mathcal{G}_2)$ and $E(\mathcal{G}) = E(\mathcal{G}_1 \times \mathcal{G}_2) \bigcup (8, 2) \setminus (6, 2)$. Therefore, $\mathcal{A}(\mathcal{G}) = \mathcal{A}(\mathcal{G}_1 \times \mathcal{G}_2) + \Delta_\mathcal{A}$ where

$$\mathcal{A}(\mathcal{G}_1) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \mathcal{A}(\mathcal{G}_2) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

and $\Delta_\mathcal{A}$ defined with $[\Delta_\mathcal{A}]_{2,8} = 1$ and $[\Delta_\mathcal{A}]_{2,6} = -1$ and zeros otherwise. The graph $\mathcal{G}$ and $\mathcal{G}_1 \times \mathcal{G}_2$ are displayed in Figure 1. The considered over $\mathcal{G}$ are (2) where $A(\mathcal{G}) = \frac{1}{|V(\mathcal{G})|} A_s(\mathcal{G}) \in \boldsymbol{A}_\otimes$.

$$B_1 = C_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, B_2 = C_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

and $\|\Delta\| = 0.026$.

IV. Kronecker Product and its Approximation

In this section, we delve into the compositional properties of the Kronecker product and approximate Kronecker product. Specifically, for the unforced dynamics, we examine the state trajectories and stability and for the forced dynamics, we discuss controllability and observability.

A. Trajectory Approximation

An attraction of the discrete dynamics associated with a Kronecker product is that they can be easily decomposed into the dynamics of its factors. Specifically, over the dynamics (1) with initial conditions $x(0) = \prod_\otimes x_i(0)$ then

$$\begin{aligned} x(k) &= A(\prod_\times \mathcal{G}_i)^k \prod_\otimes x_i(0) \\ &= \left(\prod_\otimes A(\mathcal{G}_i)\right)^k \prod_\otimes x_i(0) \\ &= \left(\prod_\otimes A(\mathcal{G}_i)^k\right) \prod_\otimes x_i(0) \\ &= \left(\prod_\otimes A(\mathcal{G}_i)^k x_i(0)\right) \\ &= \left(\prod_\otimes x_i(k)\right). \end{aligned} \tag{3}$$

Therefore, the dynamics can be run on the factor graphs then the factor trajectories can be composed together to form the full dynamics trajectory.

By extension, the Kronecker approximation of the unforced dynamics can be approximated proportional to the quality of the approximation $\|\Delta\|$. This is formalized in the following theorem.

Theorem 2. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be a pair of finite graphs and consider $x_1(k)$ and $x_2(k)$ to be states of the systems*

$$\begin{aligned} x_1(k+1) &= A(\mathcal{G}_1)x_1(k) \\ x_2(k+1) &= A(\mathcal{G}_2)x_2(k), \end{aligned}$$

where $A(\cdot) \in \boldsymbol{A}_{\otimes}$. The state trajectory generated by the dynamics

$$x(k+1) = (A(\mathcal{G}_1 \times \mathcal{G}_2) + \Delta)\, x(k) := (A_\times + \Delta)\, x(k),$$

when initialized from $x(0) = x_1(0) \otimes x_2(0)$, can be approximated as

$$x_a(k) = x_1(k) \otimes x_2(k),$$

with error bounded as $\|x(k) - x_a(k)\| \le \|\Delta\|\, \|x(0)\| \times$

$$\frac{\left\|A_\times + \frac{1}{2}\Delta + \frac{1}{2}\|\Delta\|\, I\right\|^k - \left\|A_\times + \frac{1}{2}\Delta - \frac{1}{2}\|\Delta\|\, I\right\|^k}{\left\|A_\times + \frac{1}{2}\Delta + \frac{1}{2}\|\Delta\|\, I\right\| - \left\|A_\times + \frac{1}{2}\Delta - \frac{1}{2}\|\Delta\|\, I\right\|}. \quad (4)$$

For symmetric $A_\times + \frac{1}{2}\Delta$ then $\rho\,(x(k) - x_a(k)) \le \|x(0)\| \times$

$$\left(\rho(A_\times + \frac{1}{2}\Delta) + \frac{1}{2}\rho\,(\Delta)\right)^k - \left(\rho(A_\times + \frac{1}{2}\Delta) - \frac{1}{2}\rho\,(\Delta)\right)^k. \quad (5)$$

Proof: Consider the approximate dynamics $x_a(k+1) = A_\times x_a(k)$ initialized at $x(0)$ then from equation (3)

$$x_a(k) = x_1(k) \otimes x_2(k).$$

Comparing the approximate dynamics $x_a(k)$ to $x(k)$ then

$$\begin{aligned} x(k) - x_a(k) &= \left((A_\times + \Delta)^k - A_\times^k\right) x(0) \\ \|x(k) - x_a(k)\| &= \left\|(A_\times + \Delta)^k - A_\times^k\right\| \|x(0)\|. \end{aligned} \quad (6)$$

As $-\|\Delta\|\, I \preceq \Delta \preceq \|\Delta\|\, I$ then $(A_\times + \Delta)^k - A_\times^k \preceq$

$$\left(A_\times + \frac{1}{2}\Delta + \frac{1}{2}\|\Delta\|\, I\right)^k - \left(A_\times + \frac{1}{2}\Delta - \frac{1}{2}\|\Delta\|\, I\right)^k.$$

Let $F_+ = A_\times + \frac{1}{2}\Delta + \frac{1}{2}\|\Delta\|\, I$ and $F_- = A_\times + \frac{1}{2}\Delta - \frac{1}{2}\|\Delta\|\, I$. Noting that F_+ and F_- commute then given a submultiplicative norm

$$\begin{aligned} F_+^k - F_-^k &= (F_+ - F_-)\sum_{i=0}^{k-1} F_+^{k-1-i} F_-^i \\ \left\|F_+^k - F_-^k\right\| &\le \|F_+ - F_-\| \sum_{i=0}^{k-1} \|F_+\|^{k-1-i} \|F_-\|^i \\ &= \|F_+ - F_-\| \frac{\|F_+\|^k - \|F_-\|^k}{\|F_+\| - \|F_-\|}. \end{aligned} \quad (7)$$

Combining inequality (6) and (7) leads to (4). For symmetric M, the 2-norm is the spectral radius and $\rho(M + \alpha I) = \rho(M) + \alpha$ for $\alpha \in \mathbb{R}$. Hence, as $\rho(M + \alpha I) - \rho(M - \alpha I) = 2\alpha$, inequality (5) follows. ∎

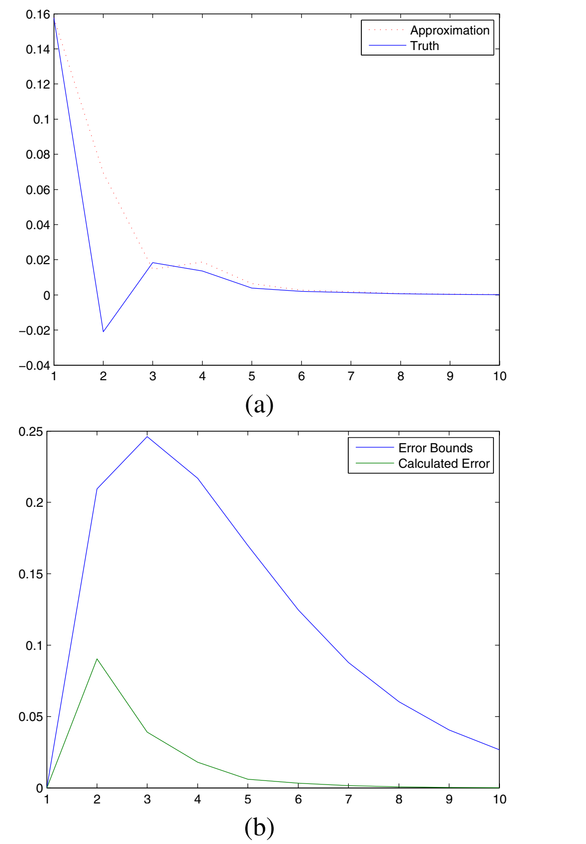

Figure 2. (a) Sample trajectory on one node. (b) True and formulated error bound from Theorem 2.

An application of the bound of Theorem 2 is illustrated in Figure 2 over the perturbed system described in Example 1.

B. Distance to Instability

In this section we show that the Kronecker factors provide the distance to instability of the Kronecker composite. An advantage of this characterization is that computationally the distance to instability of the factors is cheaper to establish compared to that of the original system. Formally, the distance to instability [30] is

$$d_A = \inf(\|\Delta\| : A + \|\Delta\| \text{ is unstable}) = 1 - \rho(d_A).$$

If $A(\cdot)$ is clear we denote $d_{A(\mathcal{G})}$ as $d_{\mathcal{G}}$.

The result is summarized in the following proposition.

Proposition 3. *For $A(\cdot) \in \boldsymbol{A}_{\otimes}$, the distance to instability $d_{\mathcal{G}_1 \times \mathcal{G}_2}$ is $d_{\mathcal{G}_1} + d_{\mathcal{G}_2} - d_{\mathcal{G}_1} d_{\mathcal{G}_2}$.*

Proof: Directly from the definition and the relationship between the singular values of the Kronecker composite and its factors,

$$\begin{aligned} d_{\mathcal{G}_1 \times \mathcal{G}_2} &= 1 - \rho(A(\mathcal{G}_1 \times \mathcal{G}_2)) \\ &= 1 - \rho(A(\mathcal{G}_1) \otimes A(\mathcal{G}_2)) \\ &= 1 - \rho(A(\mathcal{G}_1))\rho(A(\mathcal{G}_2)) \end{aligned}$$

$$= 1 - (1 - d_{\mathcal{G}_1})(1 - d_{\mathcal{G}_2})$$
$$= d_{\mathcal{G}_1} + d_{\mathcal{G}_2} - d_{\mathcal{G}_1} d_{\mathcal{G}_2}.$$

■

A consequence of Proposition 3 is that if the factors of a Kronecker product are stable then the composition system is always "more stable" in terms of the distance to instability as $d_{\mathcal{G}_1\times\mathcal{G}_2} = d_{\mathcal{G}_1} + d_{\mathcal{G}_2}\rho(A(\mathcal{G}_2)) > d_{\mathcal{G}_1}$, and similarly $d_{\mathcal{G}_1\times\mathcal{G}_2} > d_{\mathcal{G}_2}$.

An application of this is stability of a Kronecker approximation can be established if it lies within the distance to instability of its representative Kronecker composition. We illustrate this in the following example.

Example 4. Consider Example 1, the distance to instability of $A(\mathcal{G}_1)$ and $A(\mathcal{G}_2)$ are $d_{\mathcal{G}_1} = 0.62$, and $d_{\mathcal{G}_2} = 0.66$, respectively. Hence, $d_{\mathcal{G}_1\times\mathcal{G}_2} = 0.87$. Now as $\|\Delta\| = 0.026$ then $A(\mathcal{G})$ is stable and has a distance to instability of at least 0.84, in actual fact $d_{\mathcal{G}} = 0.86$.

C. Kronecker Product Controllability and Observability

The connection between the eigenspectrum of the Kronecker factors and the Kronecker composite provides a mechanism to efficiently establish controllability of a Kronecker product dynamics. Due to duality, the following results are equally applicable to observability of the pair (A, C).

The cornerstone of this analysis stems from the application of the Popov-Belevitch-Hautus (PBH) test which states that the pair (A, B) is uncontrollable if and only if there exists a left eigenvalue-eigenvector pair (λ, v) of A such that $v^T B = 0$ [31].

Core to our result is the following theorem.

Theorem 5. *[32] If $u_1, u_2, \ldots, u_n$ are linearly independent vectors and $v_1, v_2, \ldots, v_n$ are arbitrary vectors, then*

$$\sum_{k=1}^{n} u_k \otimes v_k = 0 \quad \textit{implies that} \quad v_k = 0 \quad \textit{for all } k.$$

Moreover, the roles of u_k's and v_k's in the above statement can be reversed.

The relationship between the Kronecker factor controllability and Kronecker product controllability is summarized in the following theorem.

Theorem 6. *Let $A(\cdot) \in \boldsymbol{A}_\otimes$, $\mathcal{G} = \mathcal{G}_1\times\mathcal{G}_2$, and $B = B_1\otimes B_2$. Then if $A(\mathcal{G})$ is diagonalizable, $(A(\mathcal{G}), B)$ is controllable if and only if*

(1) the pairs $(A(\mathcal{G}_1), B_1)$ and $(A(\mathcal{G}_2), B_2)$ are controllable and

(2) for $\tilde{\lambda}_1\tilde{\mu}_1 = \tilde{\lambda}_2\tilde{\mu}_2 = \cdots = \tilde{\lambda}_p\tilde{\mu}_p$, $\tilde{\lambda}_i \neq \tilde{\lambda}_j$ $\forall i \neq j$, $p > 1$,

$$B_1 \not\perp [U_1, U_2, \ldots, U_p] \ \ \textit{and/or} \ \ B_2 \not\perp [V_1, V_2, \ldots, V_p],$$

where the columns of U_i are the orthogonal left eigenvectors of eigenvalues $\tilde{\lambda}_i$ of $A(\mathcal{G}_1)$ (sim. for pairs $(\tilde{\mu}_i, V_i)$ of $A(\mathcal{G}_2)$).

Proof: As $A(\mathcal{G})$ is diagonalizable all its eigenvectors take the form $\sum u_i \otimes v_j$, where u_i and v_j are eigenvectors of $A(\mathcal{G}_1)$ and $A(\mathcal{G}_2)$, respectively. Consider an eigenvalue $A(\mathcal{G})$, say $\tilde{\lambda}_1\tilde{\mu}_1$, in terms of the set of distinct eigenvalues $\{\tilde{\lambda}_i\}$ and $\{\tilde{\mu}_i\}$ of $A(\mathcal{G}_1)$ and $A(\mathcal{G}_2)$, respectively. Then if

$$\tilde{\lambda}_1\tilde{\mu}_1 = \tilde{\lambda}_2\tilde{\mu}_2 = \cdots = \tilde{\lambda}_p\tilde{\mu}_p,$$

the left eigenvectors of $\tilde{\lambda}_1\tilde{\mu}_1$ form the basis set $\begin{bmatrix} U_1\otimes V_1 & U_2\otimes V_2 & \cdots & U_p\otimes V_p \end{bmatrix}$. Now $(A(\mathcal{G}_1\times\mathcal{G}_2), B_1\otimes B_2)$ is uncontrollable if and only if there exists an eigenvalue $\tilde{\lambda}_1\tilde{\mu}_1$ with $B_1 \otimes B_2 \perp \begin{bmatrix} U_1\otimes V_1 & U_2\otimes V_2 & \cdots & U_p\otimes V_p \end{bmatrix}$, or equivalently, there exists an eigenvector $\sum_{k=1}^{p} u_k \otimes v_k$ for some u_k in the span of U_k (sim. for v_k in V_k) such that $\sum_{k=1}^{p} u_k^T B_1 \otimes v_k^T B_2 = 0$. This occurs if and only if either (1) $B_1 \perp U_i$ and/or $B_2 \perp V_i$ for some i, i.e, $(A(\mathcal{G}_1), B_1)$ and/or $(A(\mathcal{G}_2), B_2)$ is uncontrollable or
(2) for $p > 1$, as $B_1 \not\perp U_k$ and $B_2 \not\perp V_k$ for all k, i.e., $u_i^T B_1 \neq 0$ and $v_j^T B_2 \neq 0$ for all k.

($\Longrightarrow$) From the converse of Theorem 5, (2) implies that $u_1^T B_1, u_2^T B_1, \ldots, u_p^T B_1$ are linearly dependent vectors and $v_1^T B_2, v_2^T B_2, \ldots, v_p^T B_2$ are linearly dependent vectors, or equivalently $B_1 \perp [U_1, U_2, \ldots, U_p]$ and $B_2 \perp [V_1, V_2, \ldots, V_p]$.

($\Longleftarrow$) Conversely, if $B_1 \perp [U_1, U_2, \ldots, U_p]$ then for some u in the span of $[U_1, U_2, \ldots, U_p]$ then $u^T B_1 = 0$, $(u\otimes v)^T (B_1 \otimes B_2) = 0$ for all v and so eigenvector of the composite of the form $u \otimes v$ is uncontrollable, similarly if $B_2 \perp [V_1, V_2, \ldots, V_p]$. ■

The following exercises the above theorem to establish controllability of a composite Kronecker dynamics.

Example 7. Consider Example (1), the pairs $(A(\mathcal{G}_1), B_1)$ and $(A(\mathcal{G}_2), B_2)$ are both controllable. Further, $\lambda_{1,2,3} = \{0.33, -0.083 \pm 0.22i\}$ and $\mu_{1,2,3} = \{-0.06, 0.17, 0.33\}$. Hence, as $\lambda_i\mu_j$ is unique for all i, j then condition (2) of Theorem 6 need not be checked. Hence, controllability of the factor dynamics implies directly that $(A(\mathcal{G}_1\times\mathcal{G}_2), B_1\otimes B_2)$ is controllable.

D. Distance to Uncontrollability

The previous theorem provides a method to establish controllability of a Kronecker product graph from its factors. Now the relationship to the distance to uncontrollability is explored.

The distance to uncontrollability [30] is defined as

$$d_{A,B} = \inf\{\|\Delta\| : (A+\Delta, B) \ \text{is uncontrollable}\}.$$

For brevity when $A(\cdot)$ is apparent we define $d_{A(\mathcal{G}),B}$ as $d_{\mathcal{G},B}$. The following proposition provides a mechanism to bound the distance to uncontrollability from its Kronecker products factors distance to uncontrollability.

Proposition 8. *Let $B = B_1 \otimes B_2$. The distance to uncontrollability of the pair $(A(\mathcal{G}_1\times\mathcal{G}_2), B)$ is bounded as*

$$d_{\mathcal{G}_1\times\mathcal{G}_2,B} \leq \min\left(|\lambda_1|\, d_{\mathcal{G}_2,B_2}, |\mu_1|\, d_{\mathcal{G}_1,B_1}\right),$$

where λ_1 and μ_1 are the smallest magnitude eigenvalues of $A(\mathcal{G}_1)$ and $A(\mathcal{G}_2)$, respectively.

Proof: Let u be a unit left eigenvector of $A(\mathcal{G}_1)$ associated with λ_1. Let Δ_2 corresponding to smallest uncontrollable perturbation of $(A(\mathcal{G}_2), B_2)$, i.e., $\|\Delta_2\| = d_{\mathcal{G}_2,B_2}$ and $(A(\mathcal{G}_2)+\Delta_2, B_2)$ is uncontrollable and so there exists a left eigenvector-eigenvalue pair (v,π) of $A(\mathcal{G}_2)+\Delta_2$ such that $v^T B_2 = 0$. Consider, the additive perturbation $\Delta = \lambda_1 uu^T \otimes \Delta_2$ on $A(\mathcal{G}_1 \times \mathcal{G}_2)$. Now, $u \otimes v$ is a left eigenvector of $A(\mathcal{G}_1 \times \mathcal{G}_2) + \Delta$ as

$$
\begin{aligned}
&(u \otimes v)^T \left(A(\mathcal{G}_1 \times \mathcal{G}_2) + \Delta\right) \\
&= u^T A(\mathcal{G}_1) \otimes v^T A(\mathcal{G}_2) + \lambda_1 u^T uu^T \otimes v^T \Delta_2 \\
&= \lambda_1 u^T \otimes v^T A(\mathcal{G}_2) + \lambda_1 u^T \otimes v^T \Delta_2 \\
&= \lambda_1 u^T \otimes v^T \left(A(\mathcal{G}_2) + \Delta_2\right) \\
&= \lambda_1 \pi \left(u^T \otimes v^T\right).
\end{aligned}
$$

Further, $(u \otimes v)^T B = u^T B_1 \otimes v^T B_2 = u^T B_1 \otimes 0 = 0$ and so $(A(\mathcal{G}_1 \times \mathcal{G}_2) + \Delta, B)$ is uncontrollable. Hence, $d_{\mathcal{G}_1 \times \mathcal{G}_2, B} \leq \|\Delta\| = |\lambda_1| \left\|uu^T\right\| \|\Delta_2\| = |\lambda_1| \, d_{\mathcal{G}_2,B_2}$. The proof for $d_{\mathcal{G}_1 \times \mathcal{G}_2, B} \leq |\mu_1| \, d_{\mathcal{G}_1,B_1}$ is similar. ∎

The following example demonstrates the utility of Proposition 8 providing cases where the introduced bound is loose (case (a)) and strict (case (b)).

Example 9. Consider $A(\mathcal{G}_1) = \begin{bmatrix} 4 & -1 \\ -1 & 0 \end{bmatrix}$, $B_1 = B_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$, $B = B_1 \otimes B_2$ with $\lambda_1 = 2 - \sqrt{5}$ and $d_{\mathcal{G}_1,B_1} = 1$. The following cases demonstrate the bounds in Proposition 8 are:

(a) $A(\mathcal{G}_2) = \begin{bmatrix} 0 & 2 \\ 2 & 6 \end{bmatrix}$ with $\mu_1 = 3 - \sqrt{13}$, $d_{\mathcal{G}_2,B_2} = 2$ and so $\min\left(|\mu_1| \, d_{\mathcal{G}_1,B_1}, |\lambda_1| \, d_{\mathcal{G}_2,B_2}\right) = 2\left(\sqrt{5} - 2\right) \approx 0.47$, $d_{\mathcal{G}_1 \times \mathcal{G}_2, B} \approx 0.071$.

(b) $A(\mathcal{G}_2) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ with $\mu_1 = 3$, $d_{\mathcal{G}_2,B_2} = 2$ and so $\min\left(|\mu_1| \, d_{\mathcal{G}_1,B_1}, |\lambda_1| \, d_{\mathcal{G}_2,B_2}\right) = 2\left(\sqrt{5} - 2\right) \approx 0.47$, $d_{\mathcal{G}_1 \times \mathcal{G}_2, B} \approx 0.47$.

V. Conclusion

This paper examines the properties of Kronecker product dynamics as they relate to its Kronecker factor dynamics. Results on the decomposition of trajectories, stabilizability, and controllability were explored. These results also provide properties on the approximate Kronecker product such as trajectory approximations and the distance to instability. Future work involves refining the approximate Kronecker results, especially those applicable to controllability.

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