On the Controllability and Observability of Cartesian Product Networks

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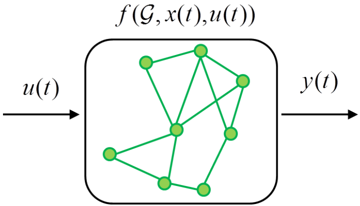
The Network in the Dynamics

General Dynamics

$$\dot{x}(t) = f(\mathcal{G}, x(t), u(t))$$
$$y(t) = g(\mathcal{G}, x(t), u(t))$$

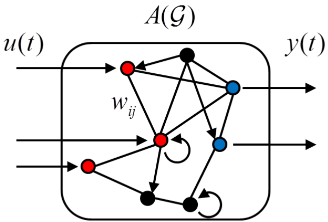


Network	System Dynamics
Graph Spectrum	Rate of convergence
Random Graphs	Stochastic Matrices
Automorphisms	Homogeneity
Graph Factorization	Decomposition

The Network in the Dynamics

Dynamics

$$\dot{x}(t) = -A(\mathcal{G})x(t) + B(S)u(t)$$
$$y(t) = C(R)x(t)$$



- First Order, Linear Time Invariant model

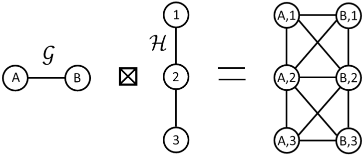
$$\dot{x}_i(t) = \left(rw_{ii} + \sum_{j \neq i} f(w_{ij}, w_{ji}) \right) x_i(t) + \sum_{j \neq i} g(w_{ij}, w_{ji}) x_j(t) + u_i(t)$$
$$y_i(t) = x_i(t),$$

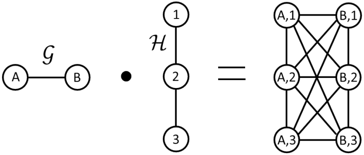
where $r \in \mathbb{R}$, $f(\cdot)$ and $g(\cdot)$ are real-valued functions, $f(0,0) = g(0,x) = 0$.

- e.g., Laplacian ($r = 0$, $f(x,y) = g(x,y) = x$), Adjacency, Advection matrices
- Input node set $S = \{v_i, v_j, \dots\}$, $B(S) = [e_i, e_j, \dots]$
- Output node set $R = \{v_p, v_q, \dots\}$, $C(R) = [e_p, e_q, \dots]^T$

Graph Products: Networks within Networks

- Many ways to compose graphs $\mathcal{G}$ and $\mathcal{H}$
  - Cartesian product $\mathcal{G} \square \mathcal{H}$
  - Tensor product $\mathcal{G} \times \mathcal{H}$
  - Strong product $\mathcal{G} \boxtimes \mathcal{H}$
  - Lexicographic product $\mathcal{G} \bullet \mathcal{H}$
  - Rooted product $\mathcal{G} \circ \mathcal{H}$
  - Corona product $\mathcal{G} \odot \mathcal{H}$
  - Star product $\mathcal{G} \star \mathcal{H}$
- How does modularity of the network manifest itself as modularity within the state dynamics?

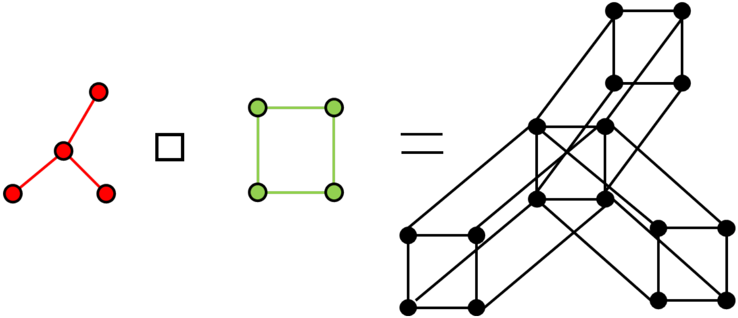




$$\text{Cartesian Product: } (\text{Graphs}, \square) \rightarrow (\text{Dynamics}, \otimes)$$

Graph Cartesian Product

- Cartesian product $\mathcal{G} \square \mathcal{H}$
- Vertex set: $V(\mathcal{G} \square \mathcal{H}) = V(\mathcal{G}) \times V(\mathcal{H})$
- Edge set: $(x_1, x_2) \sim (y_1, y_2)$ is in $\mathcal{G} \square \mathcal{H}$
    - if $x_1 \sim y_1$ and $x_2 = y_2$ or $x_1 = y_1$ and $x_2 \sim y_2$



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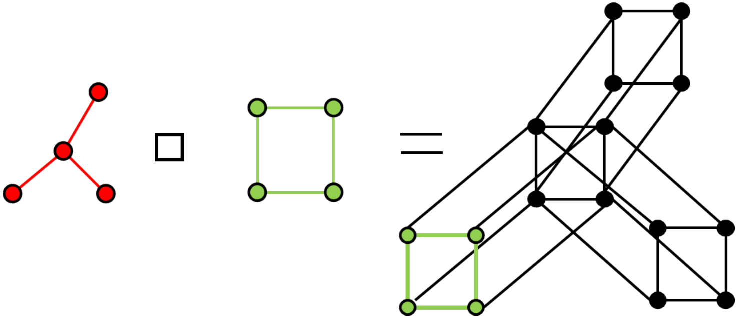

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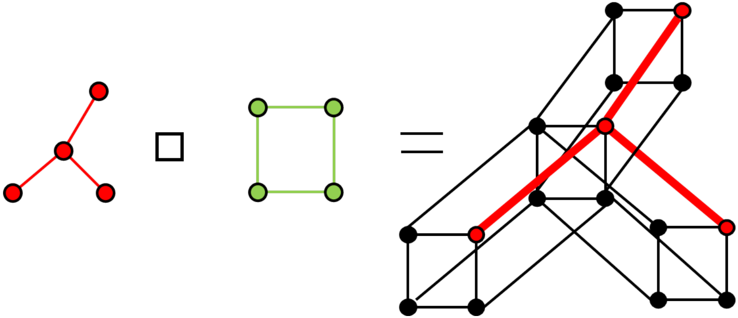
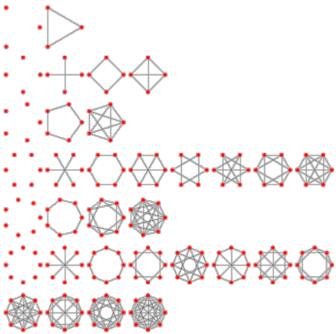
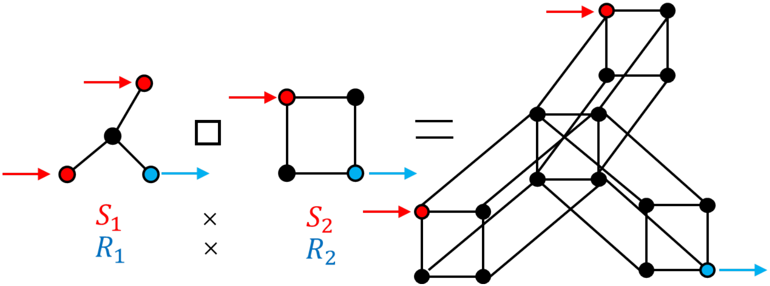


Controllability

- Dynamics are controllable if for any $x(0)$, x_f and t_f there exists an input $u(t)$ such that $x(t_f) = x_f$.
- Significant in networked robotic systems, human-swarm interaction, network security, quantum networks.
- Challenging to establish for large networks
- Known families of controllable graphs for selected inputs
    - Paths (Rahmani *et al.* '09)
    - Circulants (Nabi-Abdolyousefi *et al.* '12)
    - Grids (Parlengeli *et al.* '11)
    - Distance regular graphs (Zhang *et al.* '11)



Input and Output Set Product



Controllability Factorization - Product Control

Consider

$$\dot{x}(t) = -A(\prod_{\square} \mathcal{G}_i)x(t) + B(\prod_{\times} S_i)u(t)$$
$$y(t) = C(\prod_{\times} R_i)x(t)$$

is controllable/observable where $A(\prod_{\square} \mathcal{G}_i)$ has simple eigenvalues if and only if

$$\dot{x}_i(t) = -A(\mathcal{G}_i)x_i(t) + B(S_i)u_i(t)$$
$$y_i(t) = C(R_i)x_i(t)$$

is controllable/observable for all i.

Controllability Factorization - Idea of the Proof

Popov-Belevitch-Hautus (PBH) test

(A, B) is uncontrollable if and only if there exists a left eigenvalue-eigenvector pair (λ, v) of A such that $v^T B = 0$.

- Eigenvalue and eigenvector relationship:

	$A(\mathcal{G}_1)$	$A(\mathcal{G}_2)$	$A(\mathcal{G}_1 \square \mathcal{G}_2)$
Eigenvalue	λ_i	μ_j	$\lambda_i + \mu_j$
Eigenvector	v_i	u_j	$v_i \otimes u_j$

- Also $(v_i \otimes u_i)^T (B(S_1) \otimes B(S_2)) = v_i^T B(S_1) \otimes u_i^T B(S_2)$
- The proof follows from these observations.

Controllability Factorization - Layered Control

Consider

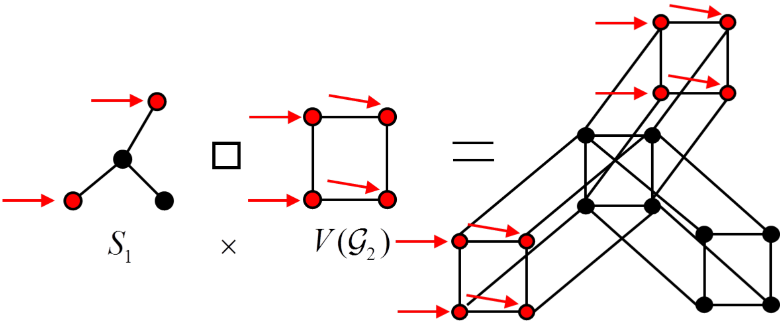
$$\dot{x}(t) = -A(\prod_{\square} \mathcal{G}_i)x(t) + B(\prod_{\times} S_i)u(t)$$
$$y(t) = C(\prod_{\times} R_i)x(t)$$

is controllable/observable where $A(\mathcal{G}_i)$'s are diagonalizable and $S_i = R_i = V(\mathcal{G}_i)$ for $i = 2, \ldots, n$ if and only if

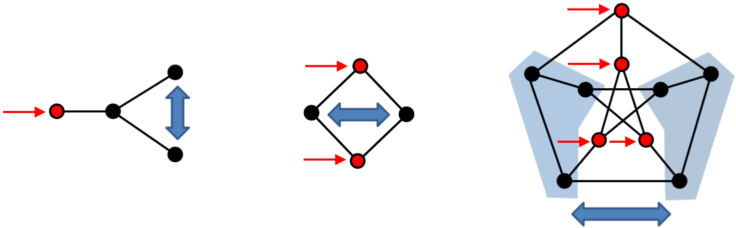
$$\dot{x}_1(t) = -A(\mathcal{G}_1)x_1(t) + B(S_1)u_1(t)$$
$$y_1(t) = C(R_1)x_1(t)$$

is controllable/observable.

Controllability Factorization - Layered Control



Uncontrollability through Symmetry



Proposition (Rahmani and Mesbahi 2006)

$(A(\mathcal{G}), B(S))$ is uncontrollable if there exists an automorphism of $\mathcal{G}$ which fixes all inputs in the set S (i.e., S is not a determining set.)

The determining number of a graph $\mathcal{G}$, denoted $Det(\mathcal{G})$, is the smallest integer r so that $\mathcal{G}$ has a determining set S of size r.

Corollary

$(A(\mathcal{G}), B(S))$ is uncontrollable if $|S| < Det(\mathcal{G})$.

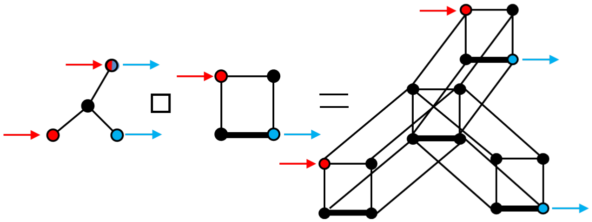
Breaking Symmetry

Automorphism group for graph Cartesian products

The automorphisms for a connected $\mathcal{G}$ is generated by the automorphisms of its prime factors.

Proposition: Smallest input set for graph Cartesian products

For controllable pairs $(A(\mathcal{G}_1), B(S_1))$ and $(A(\mathcal{G}_2), B(S_2))$ where $|S_1| = Det(\mathcal{G}_1)$ and $|S_2| = 1$. Then $S = S_1 \times S_2$ is the smallest input set such that $(A(\mathcal{G}_1 \square \mathcal{G}_2), B(S))$ is controllable.



Graph Factorization

- A graph can be factored as well as composed...

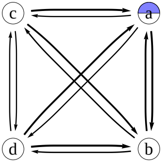
Theorem (Sabidussi 1960)

Every connected graph can be factored as a Cartesian product of prime graphs. Moreover, such a factorization is unique up to reordering of the factors.

- $\mathcal{G} = \mathcal{G}_1 \square \mathcal{G}_2$ prime implies that either $\mathcal{G}_1$ or $\mathcal{G}_2$ is K_1
  - Number of prime factors is at most $\log |\mathcal{G}|$
- Algorithms
  - Feigenbaum (1985) - $\mathcal{O}\left(|V|^{4.5}\right)$
  - Winkler (1987) - $\mathcal{O}\left(|V|^{4}\right)$ from isometrically embedding graphs by Graham and Winkler (1985)
  - Feder (1992) - $\mathcal{O}(|V||E|)$
  - Imrich and Peterin (2007) - $\mathcal{O}(|E|)$
  - C++ implementation by Hellmuth and Staude

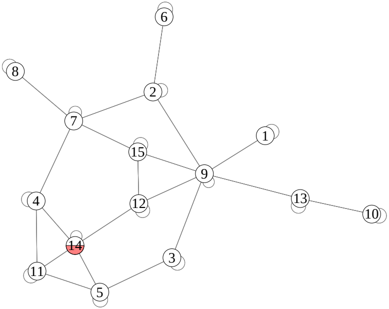
Example: Filtering on Social Product Networks

(a) Father
(b) Mother
(c) Child 1
(d) Child 2

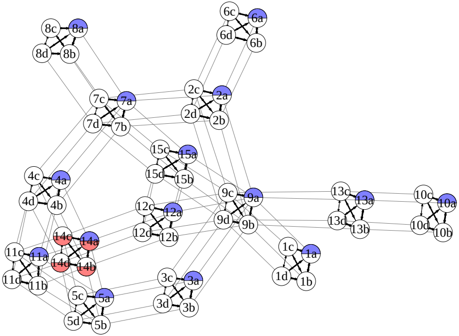


Intra-Family Network

Product Control:
$\Longrightarrow$ Father 14
Layered Control:
$\Longrightarrow$ All fathers or all of family 14



Inter-Family Network



Full Network

Conclusion

- Explored composition/factorization of dynamic network into smaller dynamic factor-networks
- Presented a factorization of controllability - a product and layered approach
- Linked the factors symmetry to smallest controllable input set
- Future work involves examining other graph products in network dynamics