A brief introduction to control theory with applications to dynamic networks

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Control Theory

- Examines the behavior of dynamic systems with inputs
- Design corrective actions (control) to acquire a desired state

Room Heating
- Heat a room to maintain a desired room temperature $T$

Aircraft Stabilization
- Maintain straight and level flight in rough weather

Network Control
- Design distributed controllers for networked systems
Ground work for robotic network control...
Control Design Process

1. Model the system (simplifying if necessary)
2. Decide what is to be controlled or observed
3. Decide on the performance specifications, or control objectives
4. Decide on the type of controller
5. Design a controller
6. Simulate and repeat from step (1) if necessary
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Dynamics Models: Room Heating

Heat input $Q$ controls the room temperature $T$

\[ \frac{d}{dt}(C_V T) = Q + \alpha(T_0 - T), \]

where $T_0$ is the outside room temperature, $C_V = 100 \times 10^3$ is the heat capacity of the room and $\alpha = 100$ is the heat transfer coefficient.

![Diagram of room heating model](image)
Dynamics Models: Aircraft Stabilization

Longitudinal dynamics

Controls: elevator angle $\delta$, Forward Acceleration $\delta_T$

States: forward speed $u$, pitch angle $\theta$, pitch rate $q$ and vertical speed $w$

\[
X = m(\dot{u} + wq) + mg \sin \theta \quad \text{(forward force)}
\]

\[
Z = \ldots \quad \text{(vertical force)}
\]

\[
M = \ldots \quad \text{(pitching moment)}
\]

where $m$ is the mass, $g$ is gravity
A multi-agent network of corresponding to node $V$ able to interact through edges $E$ defining a graph $G = (V, E)$.

- **Agreement**
  Consensus dynamics $\dot{x}_i = \sum_{\{i,j\} \in E} (x_j - x_i)$, i.e.,
  \[
  \dot{x} = -(\text{diag}(A(G))\mathbf{1} - A(G)) x = -L(G)x,
  \]
  where $A(G)$ and $L(G)$ is the adjacency and Laplacian matrix of $G$.

- **Controlled agreement**
  For inputs $S = \{k_1, \ldots, k_{|R|}\} \subseteq V$ then
  $\dot{x}_{k_l} = \sum_{\{i,j\} \in E} (x_j - x_i) + u_{k_l}$, i.e.,
  \[
  \dot{x} = -L(G)x + B(S)u, \quad y = C(R)u
  \]
  where $B(S) = [e_{k_1}, \ldots, e_{k_{|R|}}]$ and for outputs $R \subseteq V$, $C(R) = B(R)^T$.  

\[
A(\mathcal{G})
\]

\[
\begin{bmatrix}
 u(t) \\
 w_y
\end{bmatrix}
\]

\[
y(t)
\]
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Classical control
Modern control
Post-modern control
Nonlinear control
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Deriving Linear Models (SISO)

1. Formulate a nonlinear model based on physical knowledge
2. Determine a steady-state operating point about which to linearize
3. Introduce deviation variables and linearize the model

Nonlinear model: \( \frac{dx}{dt} = f(x, u), \ y = g(x, u) \)

Deviation Variables: \( \delta x(t) = x(t) - x^*, \ \delta u(t) = u(t) - u^*, \ \delta y(t) = y(t) - y^* \)

Linearize about nominal: \( (\delta x, \delta u) \)

\[
\frac{d\delta x(t)}{dt} = \left( \frac{\delta f}{\delta x} \right)^* \delta x(t) + \left( \frac{\delta f}{\delta u} \right)^* \delta u(t)
\]

\[
\delta y(t) = \left( \frac{\delta g}{\delta x} \right)^* \delta x(t) + \left( \frac{\delta g}{\delta u} \right)^* \delta u(t)
\]

State space dynamics

\[
\dot{\delta x}(t) := A\delta x(t) + B\delta u(t)
\]

\[
\delta y(t) := C\delta x(t) + D\delta u(t)
\]
Example: Room Heating Process

Dynamics:

\[
\frac{d}{dt}(C_V T) = Q + \alpha (T_0 - T),
\]

where \(C_V = 100 \times 10^3\) and \(\alpha = 100\)

Nominal: \(T^* - T_0^* = 20\), \(Q^* = 2000\)

Deviation: \(\delta T(t) = T(t) - T^*(t)\), \(\delta Q(t) = Q(t) - Q^*(t)\), \(\delta T_0(t) = T_0(t) - T_0^*(t)\)

Linearizing:

\[
\frac{d}{dt} \delta T(t) = -\frac{\alpha}{C_V} \delta T(t) + \frac{1}{C_V} \delta Q(t) + \frac{\alpha}{C_V} \delta T_0(t)
\]

\[
\dot{\delta T} = -\frac{1}{1000} \delta T + \frac{1}{100 \times 10^3} \delta Q + \frac{1}{1000} \delta T_0
\]

\[
\dot{x} = Ax + Bu + Fd, \quad y = x
\]

Matlab Example.
Laplace Transform

- \( \mathcal{L}[f(t)] = F(s) = \int_{0}^{\infty} f(t)e^{-st}dt \) where \( s = \sigma + j\omega \).
- Interpretation: A system’s response to sinusoids of varying frequency
- Represents a mathematical formulation where
  - input, output and the system are distinct and separate parts

\[
\frac{\text{Input}}{r(t)} \xrightarrow{\text{System}} \frac{\text{Output}}{c(t)}
\]

- convenient method to represent interconnections of several subsystems (cascaded interconnections)

Open-loop transfer function \( G(s) \)

For \( \dot{x}(t) := Ax(t) + Bu(t) \), \( y(t) = Cx(t) \) then

\[
Y(s) = C(sI - A)^{-1}BU(s) := G(s)U(s)
\]

- Room heating example

\[
Y(s) = X(s) = \frac{1}{100(1000s+1)}U(s) + \frac{1}{(1000s+1)}D(s) := G(s)U(s) + G_d(s)D(s)
\]
Control is proportional to error \( e = r - y_m \)

\[
u = -K(s)e = K(s)(r - y_m),
\]

where \( r \) is the reference and \( y_m = y + n \) is the measurement

- **Reason For Feedback**
  - Signal uncertainty: Unknown disturbance \( d \) and noise \( n \)
  - Model uncertainty: \( \Delta \)
  - Unstable plant

- **Block Diagram**

Matlab example.
Closed-loop System

- Ignoring noise and disturbance: \( u = -Kx \) for constant \( K \)

\[
\dot{x} = (A - BK)x, \quad y = Cx
\]

OR \( y(s) = G(s)u(s) \) and

\[
y = GK(r - y) \\
(1 + GK)y = GKr \\
y = (1 + GK)^{-1} GKr := G_{CL}r
\]

**SISO closed-loop transfer function**

\[
G_{CL} = \frac{y}{r} = \frac{GK}{1 + GK}
\]

- Why not make \( K \) as big as possible? ... stability and performance
Closed-loop Stability

- The system is **stable if and only if the closed-loop poles are in the open left-half plane (LHP)**, as

\[ G_{CL} = \frac{GK}{1 + GK} \]

equivalently, roots of \(1 + GK = 0\) or eigenvalues of \(A - BK\) are in LHP

- Example: SISO controlled agreement on 3 node path

\[ A = -L(G) = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \]

OR

\[ G(s) = \frac{1}{s(s+1)(s+3)} \]

Stable for \(K < 12\), unstable for \(K > 12\)

- Matlab Command: `sys = ss(A, B, C, 0), sisotool(sys)`
Root-Locus: Visualizing stability

- Representation of the paths of the closed-loop poles as the gain $K$ is varied

Root locus construction:

Matlab Command: $s = \text{tf('s')}, \text{rlocus}(G)$ OR $\text{sisotool()}$ again

Family of rules to generate the root locus
SISO Controlled Agreement revisited

Is there a input/output pair which are closed-loop stable for all $K > 0$? ... Yes ... When the input and output are in common

Why?

- The zeros $(z_1, \ldots, z_{n-1})$ and poles $(p_1, \ldots, p_n)$ are interlacing

\[ p_n \leq z_{n-1} \leq p_{n-1} \leq \cdots \leq z_1 < p_1 = 0 \]

Some rules: The roots locus

- begins at the finite and infinite open-loop poles and ends at the finite and infinite open-loop zeros
- exists to the left of an odd number of real-axis finite open-loop poles and/or finite open-loop zeros
Time Domain Performance

- **Rise time** \( (t_r) \): the time it takes for the output to first reach 90% of its final value
- **Settling time** \( (t_s) \): the time after which the output remains within ±2% of its final value
- **Overshoot**: the peak value divided by the final value
- **Decay ratio**: the ratio of the second and first peaks
- **Steady-state offset**: the difference between the final value and the desired final value

**Speed of Response, Quality of Response**
Second-order systems: $G(s) = \frac{1}{s^2 + \alpha s + \beta}$

$T_s = \frac{\ln 0.02}{\sigma} \approx \frac{-4}{\sigma}$

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</tr>
<tr>
<td>5%</td>
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<td>2%</td>
<td>39°</td>
</tr>
<tr>
<td>1%</td>
<td>34°</td>
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Consider the Multi-Input Multi-Output (MIMO) system

- Internal States: Forward Speed, Pitch angle, Pitch Rate, Vertical Speed
- Outputs: All
- Inputs: Elevator Angle, Forward Acceleration,

Here \( u = Ke \), \( K \in \mathbb{R}^{2 \times 4} \).

How do we design \( K \)?
Linear Quadratic Regulator (LQR)

- **Objective**: $x(0) \rightarrow x(T)$

- Optimization Approach with the formulation:

**Finite-time LQR**

\[
\min_{u = -Kx} V = \int_0^T x^T Qx + u^T Rudt + x^T(T)Mx(T) \]

s.t. $\dot{x} = Ax + Bu$,

where $Q, R, M$ are positive definite

- Large $Q$, promotes good average tracking
- Large $R$, promotes smaller control effort
- Large $M$, promotes better final time tracking
Dynamic Programming Solution

The optimal state-feedback is:

\[ u^* = -K(t)x \]
\[ K(t) = R^{-1}B^TP(t) \]

with

\[ V^*(x, t) = x^TP(t)x \]

where \( P(t) \) is the solution to the matrix Riccati equation:

\[ -\dot{P} = A^TP + PA + Q - PBR^{-1}B^TP \]
\[ P(T) = M \]

- This can be solved via a system of linear equations
- When matrices are time-invariant an explicit solution is available using the matrix exponential
Steady-State Regulator

Infinite-time LQR

\[
\min_{u=-Kx} V = \int_0^\infty x^T Q x + u^T R u dt
\]

s.t. \( \dot{x} = Ax + Bu \),

\[
0 = A^T P + PA + Q - PBR^{-1}B^T P
\]

Special Feature: \( Q = F^T F \), where \((A, F)\) is observable, \((A, B)\) is controllable, the closed loop is asymptotically stable
### Example: Aircraft Model

**Aircraft Model Parameters**

\[ A = \begin{bmatrix}
-0.0538 & -0.1712 & 0 & 0.0705 \\
0 & 0 & 1 & 0 \\
0.0485 & 0 & -0.8556 & -1.013 \\
-0.2909 & 0 & 1.0532 & -0.6859 \\
\end{bmatrix} \]

\[ B = \begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & -1.665 \\
0 & -0.0732 \\
\end{bmatrix} \]

\[ F = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{bmatrix} \]

\[ Q = F^T F \quad \text{and} \quad R = \alpha I \quad \text{(arbitrary initial} \ x(0) = \begin{bmatrix} 100 & -15 & 1 & 25 \end{bmatrix}^T) \]

**Matlab Command**

- \([K, P] = \text{lqr}(A, B, Q, R)\)
- Plotting \(\text{sys} = \text{ss}(A - BK, 0, I, 0), \text{initial}(\text{sys}, x(0))\)
Controllability and Observability

- Dynamics are **controllable** if for any \( x(0), x_f \) and \( t_f \) there exists an input \( u(t) \) such that \( x(t_f) = x_f \).
- For linear system \( \dot{x} = Ax + Bu \), closed form solutions is
  \[
  x_f(t) = e^{At}x(0) + \int_0^{t_f} e^{A(t-\tau)} Bu(\tau) d\tau
  \]
- Controllability: Does \( \mathcal{R}(e^{A(t-\tau)}B) \) spans the entire state space?
- Many methods to check this one popular one is the PBH test

**Popov-Belevitch-Hautus (PBH) test**

- The pair \((A, B)\) is uncontrollable if and only if there exists a left eigenvalue-eigenvector pair \((\lambda, v)\) of \(A\) such that \(v^T B = 0\).

- Observability is the dual of controllability hence \((A, B)\) controllable if and only if \((A, B^T)\) is **observable**
A system is **input symmetric** with respect to the input node if there exists a non-trivial automorphism that fixes all inputs, i.e., there exists a permutation matrix $J \neq I$ with

$$JA = AJ \quad \text{and} \quad BJ = B.$$ 

**Definition:** An **automorphism** of the graph $\mathcal{G}$ is a mapping $\pi : V(\mathcal{G}) \to V(\mathcal{G})$ such that if $\{i,j\} \in E(\mathcal{G}) \iff \{\pi(i), \pi(j)\} \in E(\mathcal{G})$.

- For $A = -L(\mathcal{G})$, then $JA = AJ$ implies that $J$ describes an automorphism of $\mathcal{G}$.
Proposition (Rahmani and Mesbahi 2006)

If a system \((A, B)\) is input symmetric then it is uncontrollable.

Rough Proof:
Let \(v\) be an eigenvector of \(A\) then

\[
AJv = (JA)v = J(\lambda v) = \lambda Jv
\]

So \(Jv\) is also an eigenvector, and as \(v\)'s are spanning then \(Jv \neq v\) from some \(v\). Then \(v - Jv\) is an eigenvector and

\[
(v - Jv)^T B = v^T B - v^T J^T B = v^T B - v^T B = 0
\]

and the pair \((A, B)\) is uncontrollable (PBH test).
Observer/ State Estimator

What if only some of the states are observed, i.e., \( y = Cx \) where \( C \) is not left-invertible?

Can we work out what \( x(t) \) is in real-time? ... Yes

- Open-loop State Estimator \( \dot{\hat{x}} = A\hat{x} + Bu, \hat{y} = C\hat{x} \)
- Closed-loop State Estimator \( \dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}), \hat{y} = C\hat{x} \)
- Equivalently,

\[
\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly
\]
Requoting:

\[ \dot{x} = Ax + Bu, \quad y = Cx \]
\[ \dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly = (A - LC)\hat{x} + Bu + LCx \]

Consider the error \( e(t) = x(t) - \hat{x}(t) \) then

\[ \dot{e} = \dot{x} - \dot{\hat{x}} \]
\[ = Ax + Bu - (A - LC)\hat{x} - Bu - LCx \]
\[ = A(x - \hat{x}) - LC(x - \hat{x}) \]
\[ = (A - LC)(x - \hat{x}) \]
\[ = (A - LC)e \]

Hence if \( A - LC \) is stable then \( e \to 0 \) and \( \hat{x}(t) \to x(t) \)
Controller-Estimator

Combine the controller and observer:

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
u &= -K\hat{x}
\end{align*}
\]

where \(\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly\)

- If \(L\) is selected in an optimal way, similar to LQR, the estimator is known as a Kalman filter

Nonlinear extension

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Other Design Measures: System Norms

- LQR controller often exhibit poor robustness when exposed to non-Gaussian noise and disturbances.
- Alternative measures and optimization techniques were explored.

\[ \mathcal{H}_2 \text{ norm} \]

\[ \| G(s) \|_2 = \max_{w(t) = \text{unit impulse}} \| z(t) \|_2 \]

\[ \mathcal{H}_\infty \text{ norm (induced norm)} \]

\[ \| G(s) \|_\infty = \max_{\| w(t) \|_2 = 1} \| z(t) \|_2 \]

\[ \mathcal{L}_1\text{-norm (induced norm)} \]

\[ \| g(t) \|_1 = \max_{\| w(t) \|_\infty = 1} \| z(t) \|_\infty , \]

where \( g(t) \) is the impulse response.
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Examine the nonlinear closed loop system
\[ \dot{x} = u = f(x) \] directly (without linearization)

Main method of analysis is Lyapunov theory

Examine the potential 'bowl' to steer the dynamics to the equilibrium

LaSalle's Invariance Principle

Let \( V : \mathbb{R}^n \to \mathbb{R} \) be a continuously differentiable function such that

\[ \dot{V}(x) = \frac{\partial}{\partial t} V(x) = \nabla V(x)^T f(x) \leq 0. \]

Then for \( x \in \Omega \) (a compact positively invariant set), \( x \to M \) where \( M \) is the largest invariant set contained in \( \left\{ x | \dot{V}(x) = 0 \right\} \).
Agreement dynamics $\dot{x} = -L(G)x$

- Consider the potential $V(x) = \frac{1}{2}x^T x$

- Then
  $$\dot{V}(x) = \nabla V(x)^T f(x) = x^T \dot{x} = -x^T L(G)x \leq 0.$$ 

- Selecting $\Omega = \{x \in \mathbb{R}^n | V(x) \leq c\}$, $M = \text{null}(L(G)) = \text{span} \{1\}$, 
- Therefore by LaSalle’s $x \rightarrow \text{span} \{1\}$ (the agreement subspace)

A twist for design

- Select $u \propto -\nabla V(x)$ such that $\nabla V(x)^T f(x) \leq 0$
A multi-agent network of corresponding to node $V$ able to interact through edges $E$ defining a graph $G = (V, E)$.

**Unicycle model:** Each agent is a unicycle

\[
\begin{aligned}
\dot{x}_i &= \cos \theta_i \\
\dot{y}_i &= \sin \theta_i \\
\dot{\theta}_i &= u_i
\end{aligned}
\]

Also represented in imaginary space as

\[
\begin{aligned}
\dot{r}_i &= \dot{x}_i + j \dot{y}_i = e^{j\theta_i} \\
\dot{\theta}_i &= u_i
\end{aligned}
\]

Let $e^{j\theta} = \begin{bmatrix} e^{j\theta_1} & e^{j\theta_2} & \ldots & e^{j\theta_n} \end{bmatrix}$.

Agreement on $\theta$?... Kuramoto dynamics
Potential function with minimum when $e^{j\theta} = \alpha 1$,

$$V(\theta) = \frac{1}{2} \left( e^{j\theta} \right)^* L(G) e^{j\theta}.$$ 

Then

$$-k \nabla_i V(\theta) = k j \sum_{\{i,k\} \in E} \left( e^{j(\theta_k - \theta_i)} - e^{j(\theta_i - \theta_k)} \right)$$

$$= k \sum_{\{i,k\} \in E} \sin (\theta_k - \theta_i)$$

For $\dot{\theta}_i = u_i = -k \nabla_i V(\theta) = k \sum_{j \in N(i)} \sin (\theta_j - \theta_i)$,

$$\dot{V} = -k \nabla V(x)^T \nabla V(x) \leq 0$$
Conclusion

- Classical Control - transfer function approach, with pictorial description for SISO systems
- Modern (and post-modern) Control - state space approach, heavily resting on optimization
- Controllability/Observability - the necessary number of inputs to excite and divulge the internal states
- Nonlinear control - no linearization, with a challenge to find potential function
- some others…
Glimpse of other control areas

- **Flavors of controllability**
  - traditional controllability
  - strong controllability
  - weak controllability
  - nonlinear controllability (lie brackets)

- **Compositional theory** - methods to reliably combine subsystems
  - cascade systems
  - passivity
  - Example: network of networks
Glimpse of other control areas

- Open-loop design - alter the plant for more favorable features under some measure
  - optimization
  - game theory
  - Example: network rewiring

- Others: System Identification, Switching systems, Stochastic systems...