Homework #4

Problem 3.3-4 (Uncontrolled Newton's System) Consider the system in 3.3-3. Solve the Lyapunov eq. (3.3-9) to find the cost kernel if u=0. Sketch the scalar component of S(t).

For the uncontrolled system, we have B=0:

\[
\frac{d}{dt} \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} = \begin{bmatrix} A \\ \Theta \end{bmatrix} \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix}
\quad \text{and} \quad
J = \frac{1}{2} \int_0^T x^T x \, dt \rightarrow \Theta = I, S(T) = I
\]

Solving (3.3-9), we have:

- \dot{S} = A^T S + S A + \Theta \Rightarrow S(T) = \Theta, S = \begin{bmatrix} S_1 & S_2 \\ S_2 & S_3 \end{bmatrix}

Substituting A^b, we have:

- \dot{S}_1 = 1, \quad S_1(T) = 1
- \dot{S}_2 = S_1, \quad S_2(T) = 0
- \dot{S}_3 = 2S_2 + 1, \quad S_3(T) = 1

Solving these equations gives:

- S_1(t) = T - t + 1
- S_2(t) = T - (1 + T)t + \frac{T^2}{2} + \frac{t^2}{2}
- S_3(t) = [1 + T + T^2 + \frac{t^2}{3}] - (T^2 + 2T + 1)t + (1 + T)t^2 - \frac{t^3}{3}

The cost kernel is \begin{bmatrix} S_1(t) & S_2(t) \\ S_2(t) & S_3(t) \end{bmatrix} and plots are shown below.

(could use Matlab for plotting)

Problem 3.3-5 (uncontrolled harmonic oscillator) Repeat problem 3.3-4 for the following system:

Let S(T) = I, Q = I, \omega_n^2 = 1, \delta = \frac{1}{2}

\[
\frac{d}{dt} \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\delta \omega_n \end{bmatrix} \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix}
\quad \text{and} \quad
J = \frac{1}{2} \int_0^T x^T x \, dt \rightarrow \Theta = I, S(T) = I
\]

\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} x
\]

Solving (3.3-9), Lyapunov eq. - \dot{S} = A^T S + S A + \Theta \Rightarrow S(T) = I, S = \begin{bmatrix} S_1 & S_2 \\ S_2 & S_3 \end{bmatrix}

We have:

- \dot{S}_1 = -2S_2 + 1, \quad S_1(T) = 1
- \dot{S}_2 = -S_3 + S_1 - S_2, \quad S_2(T) = 0
- \dot{S}_3 = 2S_2 - 2S_3 + 1, \quad S_3(T) = 1
We can formulate this into
\[
\begin{bmatrix}
\dot{s}_1 \\
\dot{s}_2 \\
\dot{s}_3
\end{bmatrix} =
\begin{bmatrix}
0 & 2 & 0 \\
-1 & 1 & 1 \\
0 & 2 & 2
\end{bmatrix}
\begin{bmatrix}
s_1 \\
s_2 \\
s_3
\end{bmatrix} +
\begin{bmatrix}
-1 \\
0 \\
0
\end{bmatrix}
\] or
\[
\dot{s} = \mathbf{A}s + \mathbf{B}, \quad s(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\]

And solve, we get
\[
\begin{bmatrix}
s_{1}(t) \\
s_{2}(t) \\
s_{3}(t)
\end{bmatrix} =
\begin{bmatrix}
\mathbf{S}(t) \\
\mathbf{E}(t-T) \\
\int_{t}^{T} \mathbf{E}(t-T) \mathbf{B} \mathbf{d}t
\end{bmatrix}
\]

\[
s_{1}(t) = \frac{e^{-3t}}{6} (3e^{t} + 3e^{-3}T(t-T) - 3e^{-3}T(t-T) + 3e^{3}T(t-T) + 3e^{-3}T(t-T))
\]

\[
s_{2}(t) = \frac{e^{-3t}}{6} (3e^{3}T(t-T) - 3e^{-3}T(t-T) + 3e^{3}T(t-T) - 3e^{-3}T(t-T))
\]

\[
s_{3}(t) = 1 - \frac{e^{-3}}{6} (e^{-3}T(t-T) + e^{-3}T(t-T) + e^{-3}T(t-T) - e^{-3}T(t-T))
\]

Problem 3.4-4 Suboptimal control of Newton's system. In example 3.4.3, the steady-state gain of Newton's system is of the form \( K_\infty = [\omega_n^2, 2\omega_n^2] \), for suboptimal control \( u(t) = -K_\infty x(t) \) Determine the suboptimal cost \( J_{sub} \) and compare with (3.3-3)

Substitute \( K = [\omega_n^2, 2\omega_n^2] \), \( R = I \), \( Q = \begin{bmatrix} \omega_n^4 & 0 \\ 0 & 2\omega_n^2 (2\omega_n^2) \end{bmatrix} \) into eq (3.3-5)

\[
\dot{s} = (A-BK)^T s + s(A-BK) + K^T R K + Q
\]

The ARE becomes
\[
\begin{bmatrix}
\dot{s}_1 \\
\dot{s}_2 \\
\dot{s}_3
\end{bmatrix} =
\begin{bmatrix}
-2\omega_n^2 s_1 + \omega_n^4 + q_4 \\
\omega_n^4 - 2\omega_n^2 s_2 - \omega_n^4 s_3 + 2\omega_n^4 s_4 \\
2\omega_n^4 s_1 - 4\omega_n^2 s_2 + 4\omega_n^4 s_3 + q_6
\end{bmatrix}
\]

If \( \dot{s} = 0 \) we have (solving 3 equations):
\[
\begin{align*}
s_1 &= -\omega_n^2 (\omega_n^2 + 1) \\
s_2 &= \omega_n^2 \\
s_3 &= \delta (\omega_n^2 + 1)
\end{align*}
\]

To compare with 3.3-3, we have \( Q = I \), so \( \omega_n = 1 \), \( \delta = \frac{15}{2} \) and \( s_1 = \frac{15}{2} \), \( s_2 = 1 \), \( s_3 = \frac{15}{2} \) which is the upper bound of \( s_{i}(t), s_{i}(t), s_{i}(t) \) in problem 3.3-3

\[\square\]
Problem M1: For the point mass control in 2D plane discussed in the class, design LQR that tracks \((x(t), y(t)) = (t, \sin(t))\) coordinate.

2-dim point mass is modeled as:

\[
\dot{\begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{bmatrix}} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} u = \begin{bmatrix} x(0) \\ y(0) \\ \dot{x}(0) \\ \dot{y}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0.5 \\ 0.5 \end{bmatrix} \]

You can pick this!

For reference trajectory generator, we choose:

\[
\begin{bmatrix} \dot{x}_{\text{ref}} \\ \dot{y}_{\text{ref}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{\text{ref}} \\ y_{\text{ref}} \end{bmatrix}, \quad X_{\text{ref}} = \begin{bmatrix} x_{\text{ref}}(0) \\ y_{\text{ref}}(0) \end{bmatrix}, \quad Z(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}
\]

It is noted that we can also choose \(C_z\) to be just \(I_{4x4}\) to compare both position and speed. \(Z(0)\) is chosen such that the reference mass starts its journey from origin as plots:

\[
\begin{align*}
x_{\text{ref}}(t) & = t \\
y_{\text{ref}}(t) & = \sin(t)
\end{align*}
\]

Form the system into:

\[
\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_z \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} B \end{bmatrix} u \quad \text{to min } J = \int_0^\infty (X - C_zZ)^T Q (X - C_zZ) + u^T R u \ dt
\]

We choose \(Q = q C_z\) for this case (If \(C_z = I\), we choose \(Q = q I_{4x4}\) as well), \(R = I_{2x2}\), now since

\[
(X - C_zZ)^T Q (X - C_zZ) = \begin{bmatrix} x^T & z^T \end{bmatrix} \begin{bmatrix} Q & -Q C_z \\ C_z^T Q & C_z^T Q C_z \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}
\]

The problem becomes:

\[
\min_u J = \int_0^\infty \dot{X}^T \tilde{Q} \dot{X} + u^T R u \quad \text{subject to } \dot{X} = A X + B u.
\]

Use lqr command in MATLAB then the result should be:

\[
U = -K \dot{X}
\]

(If we increase \(q\) then the trajectory should converge to the reference trajectory sooner)

\[
\begin{align*}
\text{Initial velocity is chosen to be } [\tilde{v}_x, \tilde{v}_y] \\
(x(0), y(0)) & = (1, 0) \\
(x_{\text{ref}}(1), y_{\text{ref}}(1)) & = (1, 0)
\end{align*}
\]