Today's agenda:

- infinite horizon LQR & ARE:

\[
\min_{u(t)} \int_0^\infty x^T Q x + u^T R u \, dt \quad Q \succ 0 \quad R \\
\dot{x} = Ax + Bu \quad x_0 \text{ given}
\]

Note that we assume \( A, B, Q, R \) to be time-invariant.

- An example

- Calculus of variations & optimal control (if we have time)

So the first question is: Why LQR w/ \( T \to \infty \) even interesting special case?
Example of LQR (Example 2.6)

\[ \dot{x} = Ax + Bu \]

\[
A = \begin{bmatrix}
0 & 0 & 1.132 & 0 & -1 \\
0 & -0.0538 & -0.1712 & 0 & 0.0705 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0.0485 & 0 & -0.8556 & -1.013 \\
0 & -0.2909 & 0 & 1.0532 & -0.6859 \\
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
0 & 0 & 0 \\
-0.12 & 1 & 0 \\
0 & 0 & 0 \\
4.419 & 0 & -1.655 \\
1.575 & 0 & -0.0732 \\
\end{bmatrix}, \quad C = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\end{bmatrix}
\]
The variable $x$ is the "deviation" from some trim condition:

$$x = 0 \text{ trim state}$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \end{bmatrix} = \begin{bmatrix} \text{altitude (m)} \\ \text{speed (m/s)} \\ \text{pitch (deg)} \\ \text{pitch rate (deg/s)} \\ \text{vertical speed (m/s)} \end{bmatrix}$$

It thus makes sense to design a controller to bring the airplane back to the trim if it gets perturbed from it.
control surfaces
\[ u(t) = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \text{spoiler/flap angle (deg/10)} \\ \text{forward acceleration (m/s/s)} \\ \text{elevator angle (deg)} \end{bmatrix} \]

Let us do this on Matlab and gain some insights!
Let us consider for example:

\[ \min_{u(t)} \int_0^\infty \frac{1}{2} u(t)^2 dt \]

\[ \dot{x} = x + u \quad x(0) \text{ given} \]

Then what is \( u(t) \)? well, \( u^*(t) = 0 \)!

That is not necessarily bad until we realize that the closed loop system is:

resulting

\[ \dot{x} = x \quad \text{and} \quad x(t) \to 0 \text{ if } x_0 \neq 0. \]

What is the problem:

1) original system is unstable

2) the performance index doesn't have a term that somehow encourages stable closed loop system!
You also notice that if we want finite $T^*$, $u(t) \to 0$, $Q^{1/2}x(t) \to 0$, $A x(t) \to 0$.

If $A$ & $Q$ are arbitrary, then we want $x(t) \to 0$.

But what would guarantee that there is even a control $u(t)$ s.t. $x(t) \to 0$ from an arbitrary initial condition?

Controllability of $(A, B)$!

We can in fact relax this assumption to stabilizability.
The "theorem" is as follows:

Suppose that $A, B, Q, R$ are time invariant and $T \to \infty$ in the LQR setup. Then

$$\min_{u} \int x^T Q x + u^T R u \, dt$$
$$0 \dot{x} = A x + B u$$
$x_0$ given

Then: the optimal control exists, leads to a finite cost, if

\( \mathcal{O}(A, B) \) is stabilizable (or controllable)
\( \mathcal{O}(Q, A) \) is detectable (or observable)

In this case $u^* = -R^{-1} B^T P x(t)$

where $P$ is the solution of the algebraic Riccati equation

$$0 = A^T P + PA + Q - PBR^{-1}B^T P$$
\[ x = (A - BK)x \quad \text{closed loop} \]

is stable, i.e.,

\[ A - BK \text{ is Hurwitz} \]

First, let us argue that when \( P \to \infty, P = 0 \); why?

Since \( A, B, Q, R \) are time invariant, the cost-to-go should be independent of time "t" and only a function of the state at time \( t \):

So:

\[ \text{Cost-to-go} = x(t)^T P x(t) \]

from state \( x(t) \)

a constant matrix - independent of time
You can also prove this from the solution of the first order linear matrix equation based on the Hamiltonian.

So:

\[ \text{DRE} \rightarrow \text{ARE} \rightarrow K = -RBP \]

when \( T \rightarrow \infty \)

\[ u(t) = K \times x(t) \]

Now if we want \( \Rightarrow \) closed loop dynamics

\[ \dot{x} = (A - BK)x \]

\[ \min \int_0^\infty x^T Q x + u^T R u \, dt \]

be finite, we want to make sure that \( F \in \mathbb{R}^k \)

s.t. \( (A - BK) \) is stable, i.e., \( x(t) \rightarrow 0 \)

\[ u(t) \rightarrow 0. \]
Given a pair \((A, B)\), the system \(\dot{x} = Ax + Bu\)

is called stabilizable if \(\exists K\) s.t.

\[ A - BK \text{ is Hurwitz.} \]

On the other hand, \((A, B)\) is called controllable
if the eigenvalues of \(A + BK\) can be arbitrary assigned.

So a controllable system is always stabilizable
but not vice versa, \((\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, [0])\) is stabilizable but not controllable.
So it is more clear now why finiteness of
\[ J^* = \int_0^\infty x^TQx + u^TRu \, dt \]
\[ x^* = Ax + Bu \quad x_0 \text{ given (arbitrary!)} \]
depends on
controllability/stabilizability of
\[ (A, B) \]

Thus for \((A, B), \exists K \text{ s.t. } (A - BK) \text{ is stable} \) but why should this \( K \) be
the one obtained from the LQR?!
That is where we need another requirement: if \( Q = DD^T \), say \( Q = Q^{1/2} Q^{1/2} \),

we require that

\[(Q^{1/2}, A) \text{ is observable or} \]
\[(D, A) \text{ is detectable.} \]

Typically we write

"(A, B) is controllable"

and "(C, A) is observable" (if you write "(A, C) is observable it is ok!")
That is, we claim that if \((Q^2, A)\) is observable/detectable, then the solution of LQR leads to a stabilizing feedback!

To show this, yes, we need some nice linear algebra; we will focus on observability (the extension to detectable case becomes obvious).

What is **observable** observability?

\[
\begin{align*}
\dot{x} &= Ax \\
y &= D e^{At} x_0
\end{align*}
\]
the pair \((D, A)\) is observable if

\[ y(t) = 0 \quad \forall t \Rightarrow x_0 = 0. \]

We should be able to claim that having seen a zero signal implies that \(x_0\) has to be zero. Okay, so let us prove that the LQR solution is stabilizing.

We need a few preliminary observations first:

1. If \([A, D]\) is observable and

\[
[D, A] := A'P + PA = -D'D
\]

then \(A\) is Hurwitz (stable)
why? Let $A x = \lambda x$ be an eigenvalue of $A$ $x \neq 0$.

1. Consider $x^*(A^T P + PA)x = -x^* D^T D x$

$\Rightarrow (A x)^* P x + x^* P A x = -x^* D^T D x$

$\Rightarrow x^* (A x)^* P x + x^* P (A x) = -x^* D^T D x$

$\Rightarrow x^* (x^* P x) + \lambda x^* p x = -x^* D^T D x$

$\Rightarrow \lambda (x^* P x) + \lambda x^* p x > 0$

$\Rightarrow 2 \text{Re}(\lambda) = -\|Dx\|^2$

so $\text{Re}(\lambda) < 0$ if $Dx \neq 0$. 

$\|y\|^2 = y^* y$

if $y$ is complex $y^* y = \|y\|^2$.
well if $x$ is an eigenvector of $A$, observability of $(D, A)$ ensures that $Dx \neq 0$.

otherwise $x \neq 0$, $Ax = Jx$

$A^2x = J^2x$

$Ae^x = e^{Jx}$

$e^{At}x = e^{Jt}x$

$D e^{At}x = e^{Jt}Dx = 0$.

therefore

$A$ is Hurwitz (all eigenvalues have negative real parts)

if $A^TP + PA = -DD^TP > 0$. 
what if we had something extra:

\[ A^T P + PA = -D^T D W \quad \text{and} \quad W \succ 0 \]

and \((D, A)\) is observable.

Then you can follow the same algebraic steps to show that

\[ \text{Re} (\lambda) < 0 \quad \text{where} \quad \lambda \text{ is an eigenvalue of } A. \]

Okay - we are now ready to tackle the main result! First a lemma:

Suppose that \(P\) is the solution of ARE:

\[ A^T P + PA + Q - P B R^{-1} B^T P = 0 \quad (Q = D^T D) \]

(we know that \(P \succ 0\)). Then if \((D, A)\) is observable, \(P \succ 0\) (positive definite). In fact,
If $P > 0$, $(D, A)$ has to be observable.

Proof: we know that $P \succ 0$ (cost-to-go argument)

Suppose that $\exists \ x_0 \neq 0$ s.t. $x_0^T P x_0 = 0$ (if $P$ is positive definite, $\not\exists \ x_0 \neq 0$ s.t. $x_0^T P x_0 = 0$)

If $x_0^T P x_0 = 0 = \int_0^\infty x^T Q x + u^T R u \ dt$

starting from $x_0$, then $v(t) = 0$, for all $t$.

$\Rightarrow x(t) = e^{At} x_0 \Rightarrow \int_0^\infty x_0 e^{At} DD e^{At} x_0 \ dt = 0.$

$\Rightarrow \int_0^\infty \| D e^{At} x_0 \|^2 \ dt = 0 \Rightarrow D e^{At} x_0 = 0$

$\&$ since $(D, A)$ is observable, $x_0 = 0!$. But this is a contradiction.
On the other hand, if \( \exists x_0 \) s.t. \( \text{De}^{At} x_0 = 0 \) for all \( t \), we can let \( u = 0 \).

\[
\Rightarrow J^* = x_0^T P x_0 = \int_0^\infty x(t)^T Q x(t) + u^T R u \, dt \\
= \int_0^\infty \| \text{De}^{At} x_0 \|^2 \, dt.
\]

\[
= 0.
\]

\( P \) is not positive definite since \( \exists x_0 \) s.t. \( x_0^T P x_0 = 0 \).
We are almost there...

LQR solution: \( u = -R^{-1}B^TPx \)

\[ \Rightarrow \text{closed loop} \quad \dot{x} = Ax - BRB^TPx \]
\[ = (A - BRB^TP)x \]
\[ = \overbrace{(A - BRB^TP)x}^{A_c} \]

Consider
\[ A_c^T P + PA_c = (A - BRB^TP)^T P + P (A - BRB^TP) \]
\[ = A^T P + PA - PBRB^TP - PBR^TBP - Q \]

\[ Q = DD^T \]

\[ = -DD^T - PBRB^TP = -DD^T - PBRR^TBP^{-1} - K \]
\[ = -DD^T - KRK \]
\[ \text{positive semi-def.} \]
So $A_c$ is Hermitian!

This is awesome but you can also see this from the cost function $u$ we want $x \to 0$ so any nonzero $x$ should show up in the cost! There is actually one other step that is needed here & that is if

$$(D,A)$$ is observable

is it true that $$(D,A_c)$$ is observable?

The answer is yes! Let us see why...