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UAV Flocking with Wind Gusts: Adaptive Topology and Model Reduction

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***Abstract*—In this paper, we examine the problem of UAV flocking in the presence of wind gusts. Firstly, we model a velocity consensus-based leader-follower system exposed to gust disturbances and design an optimal controller, in the linear quadratic sense, to improve velocity tracking. We then proceed to examine topological features that promote the performance of such optimal controllers to design a network rewiring protocol for improved system performance. Finally we present a novel partitioning schemed, dubbed *leader partition*, in order to fuse "similar" states in the network, forming a graph theoretic method for model reduction.**

***Index Terms*—Leader-Follower consensus; UAV flocking; Consensus protocol; Model reduction; Adaptive graphs; Coordinated control over networks**

I. Introduction

Consensus provides a framework for simple but effective distributed information-sharing and control for networked, multi-agent systems in settings such as multi-vehicle control, formation control, swarming, and distributed estimation; see for example, [1], [2]. One of the popular adaptions of traditional consensus is leader-follower dynamics [3], where leader agents within the network can control the network by exploiting the other agents' consensus dynamics and network topology. In this paper, we investigate the relationship between the network topology and the modeling and controller design of a leader-follower system. In this work, an LQG controller is applied to a leader-follower model and topological features that improve controller performance, measured in terms of the trace of the controllability gramian, are employed to rewire the network to improve performance. Finally, a network topology based model reduction is proposed that takes advantage of symmetries or near-symmetries in the network.

UAV flocking, the case study of this paper, is an example of a leader-follower model and involves the distribution of tasks, normally performed by one central aerial vehicle, to many smaller vehicles which are coordinated by leaders among the flock. One of the costs of such an architecture is increased susceptibility to external disturbances such as wind gusts. We model the UAV flocking with leader-follower dynamics running a consensus-based protocol with the objective of reaching agreement on their velocity. In the meantime, in light of the presence of wind gusts, an LQG controller is implemented on the leader agents to reject the effect of wind gust disturbance on the overall network.

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The concept of designing topologies to optimize for certain metrics has been addressed in [4] for maximizing the second smallest eigenvalue of the graph Laplacian, in [5] for optimizing the network $\mathcal{H}_2$ performance, and in [6] for maximizing the largest eigenvalue of the graph Laplacian, each using optimization techniques over *weighted* graphs. Intuitive based methods of network reconfiguration have been designed to improve network resilience, for example using thresholding methods to decide when to alter the topology [7]. Our approach is to perform edge trades to optimize the trace of the controllability gramian. In an *optimization setting* this would require NP-hard mixed-integer programming.

Traditional methods of model reduction such as balanced realization, balanced residualization and Hankel norm approximations [8] are purely system-based with minimal, if any, network interpretation. The significance of network symmetry and its role in controllability have recently been investigated [2], [9], and provides a natural extension of system based to network structure based model reduction. Preliminary work in this direction was undertaken by [10] using single leader-follower systems. We extend the approach for multi-leader scenarios.

The paper is organized as follows. §II contains the problem formulation and relevant background. A UAV flock exposed to wind gusts is introduced and an LQG controller is designed for disturbance rejection. An analysis of the trace of the controllability gramian is presented in §III, its relationship to the effective resistance is established and subsequently used to design a network rewiring protocol to improve LQG performance. §IV presents a network partitioning technique that can be used for fusing nodes to form a reduced model system but without losing the graph theoretic interpretation of the dynamics. We conclude the paper with a few remarks in §VI.

II. Background and Model

We provide a brief background on the models that will be used in this paper, including abbreviated descriptions on graphs and the consensus protocol in its controlled versions. First we introduce the notation: $\|\cdot\|_2$ and $\|\cdot\|_\infty$ denote the Euclidean and infinity norms respectively; $\mathbf{tr}(\cdot)$ denotes the trace of a matrix; $|\cdot|$ denotes the cardinality of a set; $\preceq$ denotes positive semidefinite ordering of matrices; $\mathbf{1} := [1, \ldots, 1]^T$.

An undirected graph $\mathcal{G} = (V, E)$ is defined by a node set V with cardinality n, the number of nodes in the graph, and an edge set E comprised of pairs of nodes, where nodes v_i and v_j are adjacent if $\{v_i, v_j\} \in E \subseteq [V]^2$.[1] A special

[1]The notation $[V]^2$ refers to the set of two-element subsets of V.

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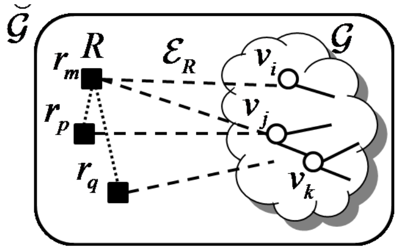

Fig. 1. Example of leader-follower notation.

family of graphs is tree graphs $\mathcal{T}$ where all two node pairs are connected by exactly one simple path.

We denote the set of nodes adjacent to v_i as $\mathcal{N}(v_i)$ and the minimum path length, induced by the graph $\mathcal{G}$, between nodes v_i and v_j as $d(v_i, v_j)$. The degree δ_i of node v_i is the number of its adjacent nodes. The degree matrix $\Delta(\mathcal{G}) \in \mathbb{R}^{n \times n}$ is a diagonal matrix with δ_i at position (i, i). The adjacency matrix is a $n \times n$ symmetric matrix with $[\mathcal{A}(\mathcal{G})]_{ij} = 1$ when $\{v_i, v_j\} \in E$ and $[\mathcal{A}(\mathcal{G})]_{ij} = 0$ otherwise. The combinatorial Laplacian is defined as $L(\mathcal{G}) = \Delta(\mathcal{G}) - \mathcal{A}(\mathcal{G}) \in \mathbb{R}^{n \times n}$ which is a (symmetric) positive semi-definite matrix.

Now consider $x_i(t) \in \mathbb{R}$ to be the i-th node's (or for our case agent's) state at time t. The continuous-time consensus protocol is defined as $\dot{x}_i(t) = \sum_{\{i,j\} \in E} (x_j(t) - x_i(t))$. In a compact form with $x(t) \in \mathbb{R}^n$, the collective dynamics is represented as $\dot{x}(t) = -L(\mathcal{G})x(t)$ with $L(\mathcal{G})$ being the Laplacian of the underlying interaction topology [1].

We next introduce a model for leader-follower consensus over a graph $\breve{\mathcal{G}} = (\breve{V}, \breve{E})$ associated with a pair $\mathcal{R} = (R, \mathcal{E}_R)$, where $R \in \breve{V}$ is the cardinality r leader agent set and $\mathcal{E}_R \subseteq \breve{E}$ is the set of edges used by the leader agents to inject signals into the network. It is assumed that for a leader agent $r_j \in R$ the same signal $u_j(t) \in \mathbb{R}$ is delivered along every edge adjacent to it. The remaining edges and agents of $\breve{\mathcal{G}}$ form the subgraph $\mathcal{G}$, with the exception of those edges between leaders which are removed. Figure 1 provides a graphical representation of this notation and setup.

The resulting leader-follower system now assumes the form,

$$\dot{x}(t) = A(\mathcal{G}, \mathcal{R})x(t) + B(\mathcal{R})u(t) := f(x(t), u(t)), \qquad (1)$$

where $B(\mathcal{R}) \in \mathbb{R}^{n \times r}$ with $[B(\mathcal{R})]_{ij} = 1$ when $\{r_j, v_i\} \in \mathcal{E}_R$ and $[B(\mathcal{R})]_{ij} = 0$ otherwise, and

$$A(\mathcal{G}, \mathcal{R}) := -\left(L(\mathcal{G}) + M(\mathcal{R})\right) \in \mathbb{R}^{n \times n}, \qquad (2)$$

where $M(\mathcal{R}) \in \mathbb{R}^{n \times n}$ with $[M(\mathcal{R})]_{ii} = \delta_i^r$, where δ_i^r is the number of leaders adjacent to v_i and $[M(\mathcal{R})]_{ij} = 0$ otherwise. We define δ_j^v as the number of non-leader agent adjacent to r_j. We distinguish two special cases of this setup; one in which there is exactly one leader for each edge $\mathcal{E}_R$ and so a *distinct* control signal is delivered through each edge, denoted with the leader pair $\mathcal{R}_d$, and one where there exists only one leader node so a *common* signal is delivered through each edge, denoted with pair $\mathcal{R}_c$. We also denote the set of agents v_i such that $\{r_j, v_i\} \in \mathcal{E}_R$ by $\pi(\mathcal{E}_R)$; this is the set of agents that directly connect to leader agents.

We recognize $A(\mathcal{G}, \mathcal{R})$ in (2) as the Dirichlet matrix, or grounded Laplacian [11]. The spectrum of $A(\mathcal{G}, \mathcal{R})$ relates closely to the spectrum of $L(\mathcal{G})$. In this way, the structure of the underlying graph is related to the dynamics of model (1). It is known, for example, that the matrix $A(\mathcal{G}, \mathcal{R})$ of model (1) is negative definite (and so invertible) if the original graph is connected [12]. In the next section we proceed to disturb the leader-follower system with a wind gust.

A. Wind Model

A vertical wind gust w_g is not white, but has a power spectral density given in Dryden form [13] as

$$\Phi_w(\omega) = 2L\sigma^2 \frac{1 + 3L^2\omega^2}{(1 + L^2\omega^2)^2}, \qquad (3)$$

with w the frequency in rad/s, σ the turbulence intensity, and L the turbulence scale length divided by true airspeed.

The power spectral density (3) can be factored as $\Phi_w(s) = H_w(s)H_w(-s)$, where [13]

$$H_w(s) = \sigma\sqrt{\frac{6}{L}} \frac{s + 1/L\sqrt{3}}{s^2 + 2s/L + 1/L^2}.$$

A realization of $H_w(s)$ is

$$\begin{aligned} \dot{z} &= \begin{bmatrix} 0 & 1 \\ -\frac{1}{L^2} & -\frac{2}{L} \end{bmatrix} z + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w := A_w z + B_w w \\ w_g &= \gamma \begin{bmatrix} \frac{1}{L\sqrt{3}} & 1 \end{bmatrix} z := C_w z, \end{aligned}$$

where w is white noise input with zero mean and unit variance W and $\gamma := \sigma\sqrt{6/L}$.

We model the effect of the gust on vehicle i as $\dot{x}_i = f_i(x, u) + h_i w_g$, where $h_i \in \mathbb{R}$ is specific to agent i's gust-vehicle interaction.

Assume that the same gust acts upon all agents in the network and that each follower agent is modeled as a single integrator and has adopted a consensus algorithm for velocity alignment (x state) as in (1). Therefore $f_i(x(t), u(t)) = [f(x(t), u(t))]_i$ and the full dynamics are

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} &= \begin{bmatrix} f(x(t), u(t)) \\ \dot{z} \end{bmatrix} + \begin{bmatrix} H \\ 0 \end{bmatrix} w_g \\ &= \begin{bmatrix} A(\mathcal{G}, \mathcal{R})x + B(\mathcal{R})u \\ A_w z + B_w w \end{bmatrix} + \begin{bmatrix} H \\ 0 \end{bmatrix} C_w z \\ &= \begin{bmatrix} A(\mathcal{G}, \mathcal{R}) & A_z \\ 0 & A_w \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B(\mathcal{R}) \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ B_w \end{bmatrix} w \\ &:= \mathcal{A} \begin{bmatrix} x \\ z \end{bmatrix} + \mathcal{B}u + Gw \end{aligned} \qquad (4)$$

where $H = [h_1, \ldots, h_n]^T$ and $A_z = HC_w$.

B. LQG Controller

We now investigate controlling the vehicle velocities of the wind gust disturbed system (4) using an LQG control framework. We assume that the agent states of the system can be sensed with some measurement noise v which is uncorrelated zero-mean, Gaussian, white noise random vector with correlation matrix V, as such $y = \mathcal{C} \begin{bmatrix} x \\ z \end{bmatrix} + v$, where $\mathcal{C} = \begin{bmatrix} I_{n \times n} & 0_{n \times 2} \end{bmatrix}$.

As $A(\mathcal{G},\mathcal{R})$ is negative definite it has only negative eigenvalues. Examining A_w, $\lambda_i(A_w) = -\frac{1}{L}$ for $i = 1, 2$ and as $L > 0$, it follows that A_w has only negative eigenvalues. Therefore the state matrix $\mathcal{A}$ is stabilizable and detectable so a stabilizing compensator can be formed. Hence given a performance measure

$$J = \lim_{t\to\infty} \mathbb{E}\left\{x^T Q x + u^T R u\right\},$$

a compensator that minimizes J with optimal performance cost

$$J^* = \mathbf{tr}(P K_f V K_f^T) + \mathbf{tr}(SQ), \tag{5}$$

with $K_c = R^{-1}\mathcal{B}^T P$ and $K_f = S\mathcal{C}^T V^{-1}$, the compensator and filter gains, and P and S obtained by solving appropriate algebraic Riccati equations [8].

We now present a protocol to supplement the agent dynamics control of LQG with an edge trading, adaptive topology controller to improve this nominal LQG performance.

III. Adaptive Topology Controller

As a group of networked UAVs do not require physical interconnections for their coordinated behavior, they have the advantage that their inter-vehicle communications can be rewired. This observation leads us into our next form of gust correction, namely via an *adaptive topology controller*. We present a system-theoretic metric, that can be exploited to adapt the network topology with the objective of improving the nominal LQG performance. The controllability gramian, defined as $P_T(\mathcal{A},\mathcal{B}) := \int_0^T e^{\mathcal{A}\tau}\mathcal{B}\mathcal{B}^T e^{\mathcal{A}^T\tau} d\tau$ for system (4), proves to be particularly suitable for such an analysis, and specifically, $\mathbf{tr}(P_\infty(\mathcal{A},\mathcal{B}))$ as our scalar metric motivated by the following observations:

a) The $\mathcal{H}_2$ norm for our system is

$$\begin{aligned}\|G(s)\|_2 &= \sqrt{\mathbf{tr}\left(\mathcal{C}P_\infty\left(\mathcal{A},\mathcal{B}\right)\mathcal{C}^T\right)} \\ &= \sqrt{\mathbf{tr}\left(P_\infty\left(A(\mathcal{G},\mathcal{R}),B(\mathcal{R})\right)\right)},\end{aligned}$$

where the state-space realization is $G(s) = \mathcal{C}\left(sI - \mathcal{A}\right)^{-1}\mathcal{B}$.

b) The energy of the states at the output from a unit impulse input u when $x(0) = 0$ is

$$\begin{aligned}\int_0^\infty x(t)^T x(t) dt &= \sum_{i=1}^m \|z_i(t)\|_2^2 \\ &= \mathbf{tr}(P_\infty(A(\mathcal{G},\mathcal{R}),B(\mathcal{R}))),\end{aligned} \tag{6}$$

where the $z_i(t)$'s are the output vectors resulting from applying a unit impulse along each of the orthogonal bases of the input space $\mathbb{R}^m$.

We note that P will be dependent on $\mathcal{G}$ and $\mathcal{R}$, and as we will be focusing on the steady state case, i.e., $T \to \infty$, we henceforth denote $P_\infty(A(\mathcal{G},\mathcal{R}),B(\mathcal{R})) = P(\mathcal{G},\mathcal{R})$.

We proceed to analyze this metric for our two special leader agent cases $\mathcal{R}_c$ and $\mathcal{R}_d$.

Proposition 1: For a connected graph $\mathcal{G}$, and a common signal delivered by all leaders then

$$\mathbf{tr}(P(\mathcal{G},\mathcal{R}_c)) = \frac{1}{2}\left|\mathcal{E}_R\right|.$$

Proof: As the leaders are all delivering a common signal is is equivalent to fusing the leaders together, subsequently we have $B(\mathcal{R}_c) = B(\mathcal{R}_d)\mathbf{1}$. Further, as $A(\mathcal{G},\mathcal{R})$ is only dependent on $\mathcal{G}$ and δ_i^r then $A(\mathcal{G},\mathcal{R}_c) = A(\mathcal{G},\mathcal{R}_d)$ and so $A(\mathcal{G},\mathcal{R}_c)^{-1}B(\mathcal{R}_c) = A(\mathcal{G},\mathcal{R}_d)^{-1}B(\mathcal{R}_d)\mathbf{1} = -\mathbf{1}$ hence

$$\begin{aligned}\mathbf{tr}(P(\mathcal{G},\mathcal{R}_c)) &= \mathbf{tr}(B(\mathcal{R}_c)^T \int_0^\infty e^{2A(\mathcal{G},\mathcal{R}_c)\tau} d\tau B(\mathcal{R}_c)) \\ &= -\frac{1}{2}\mathbf{tr}(B(\mathcal{R}_c)^T A(\mathcal{G},\mathcal{R}_c)^{-1}B(\mathcal{R}_c)) \\ &= \frac{1}{2}\mathbf{tr}(\mathbf{1}^T B(\mathcal{R}_d)^T\mathbf{1}) = \frac{1}{2}\left|\mathcal{E}_R\right|.\end{aligned}$$

■

Remark 1: The implication of Proposition 1 is that for the case of a single leader then $\mathbf{tr}(P(\mathcal{G},\mathcal{R})) = \frac{1}{2}\delta(r_1)$, so selecting the agent within the network with the highest degree will maximize the $\mathcal{H}_2$ norm of the system, regardless of the structure of the network.

Further, it has previously been established that the diagonal of $-A(\mathcal{G},\mathcal{R}_d)^{-1}$ where $A(\mathcal{G},\mathcal{R}_d)$, has a resistive electrical network interpretation [11]. In this setup, the agents V and R, defined in §II, represent connection points between resistors corresponding to the edges E and $\mathcal{E}_R$. In addition, all connection points corresponding to the set R are electrically shorted. The effective resistance between two connection points in an electrical network is defined as the potential drop between the two points, when a 1 Amp current source is connected across the two points. The i-th diagonal element of $-A(\mathcal{G},\mathcal{R}_d)^{-1}$ is the effective resistance $E_{\text{eff}}(v_i)$ between the common shorted external agents R and v_i. It has previously been shown [12] that $\mathbf{tr}(P(\mathcal{G},\mathcal{R}_d)) = -\frac{1}{2}\mathbf{tr}\left(M\left(\mathcal{R}_d\right)A(\mathcal{G},\mathcal{R}_d)^{-1}\right) = \frac{1}{2}\sum_{v_i\in\pi(\mathcal{E}_R)} E_{\text{eff}}\left(v_i\right)$.

We now relate the controllability gramian of a generic $\mathcal{R}$ with control signal $u \in \mathbb{R}^{|R|}$ and $\mathcal{R}_c$ and $\mathcal{R}_d$ with control signals $u_c \in \mathbb{R}$ and $u_d \in \mathbb{R}^{|\mathcal{E}_R|}$, respectively. We design these special leader sets by fusing all the leaders in R (in $\mathcal{R}$) to form $\mathcal{R}_c$ (and thus a common control signal is sent along all $\mathcal{E}_R$) and designating a distinct leader for each $\mathcal{E}_R$ in $\mathcal{R}$ to form $\mathcal{R}_d$. Consequently, $B(\mathcal{R}_c) = B\left(\mathcal{R}\right)\mathbf{1}$ and $B\left(\mathcal{R}\right) = H_1 B(\mathcal{R}_d) H_2$ where $H_1 \in \mathbb{R}^{n\times n}$ is diagonal with $\left[H_1\right]_i = \sqrt{\delta_i^r}$, and $H_2 \in \mathbb{R}^{|R_d|\times|R|}$ with $[H_2]_{ij} = \frac{1}{\sqrt{\delta_j^v}}$ if $(r_j, v_i) \in \mathcal{E}_R$ and $[H_2]_{ij} = 0$. As $A(\mathcal{G},\mathcal{R})$ is only dependent on $\mathcal{G}$ and δ_i^r then $A(\mathcal{G},\mathcal{R}) = A(\mathcal{G},\mathcal{R}_c) = A(\mathcal{G},\mathcal{R}_d)$.

We can now use these properties to bound $\mathbf{tr}(P(\mathcal{G},\mathcal{R}))$ for a generic $\mathcal{R}$.

Lemma 2: For a graph $\mathcal{G}$ and leader set $\mathcal{R}$,

$$\frac{1}{2\left|R\right|}\left|\mathcal{E}_R\right| \leq \mathbf{tr}(P(\mathcal{G},\mathcal{R})) \leq \frac{1}{2}\sum_{v_i\in\pi(\mathcal{E}_R)} \alpha_i E_{\text{eff}}\left(v_i\right),$$

where $\alpha_i = \sum_{(r_j,v_i)\in\mathcal{E}_R} \delta_j^v$, i.e., α_i is the sum the non-leader degree for each leader attached to v_i.

Proof: For a impulse applied along the basis $\frac{1}{\sqrt{|R|}}\mathbf{1}$ of $\mathbb{R}^{|R|}$ let the impulse response applied to system $\left(A\left(\mathcal{G},\mathcal{R}\right), B\left(\mathcal{R}\right)\right)$ be $z_1(t)$ and the remaining bases have impulse responses $z_2(t),\ldots,z_{|R|}(t)$. Therefore, $\mathbf{tr}(P_\infty(A(\mathcal{G},\mathcal{R}),\frac{1}{\sqrt{|R|}}B(\mathcal{R})\mathbf{1})) = \|z_1(t)\|_2^2 \leq$

$\sum_{i=1}^{|R|} \|z_i(t)\|_2^2 = \mathbf{tr}\left(P(\mathcal{G},\mathcal{R})\right)$ from (6). Similarly, $\mathbf{tr}\left(P(\mathcal{G},\mathcal{R})\right) = \mathbf{tr}(P_\infty(A(\mathcal{G},\mathcal{R}), H_1 B(\mathcal{R}_d) H_2)) \leq \mathbf{tr}(P_\infty(A(\mathcal{G},\mathcal{R}), H_1 B(\mathcal{R}_d)))$.

Now,

$$\mathbf{tr}(P_\infty(A(\mathcal{G},\mathcal{R}), \frac{1}{\sqrt{|R|}} B(\mathcal{R})\mathbf{1})) = \frac{1}{|R|}\mathbf{tr}(P(\mathcal{G},\mathcal{R}_c)).$$

Applying Proposition 1 the lower bound follows.

Similar to the proof of Proposition 1, $\mathbf{tr}(P_\infty(A(\mathcal{G},\mathcal{R}), H_1 B(\mathcal{R}_d))) = -\frac{1}{2}\mathbf{tr}\left(B(\mathcal{R}_d) H_1 H_1^T B(\mathcal{R}_d)^T A(\mathcal{G},\mathcal{R}_d)^{-1}\right)$. Let $M := B(\mathcal{R}_d) H_1 H_1^T B(\mathcal{R}_d)^T$, we note that M is a diagonal matrix with $[M]_{ii} = \delta_j^v$ if $(r_j, v_i) \in \mathcal{E}_R$ and $[M]_{ii} = 0$, otherwise and so

$$\begin{aligned}&\left[M A(\mathcal{G},\mathcal{R}_d)^{-1}\right]_{ii} \\ &= \begin{cases} \alpha_i \left[A(\mathcal{G},\mathcal{R}_d)^{-1}\right]_{ii} = -\alpha_i E_{\text{eff}}(v_i) & \text{if } v_i \in \pi(\mathcal{E}_R) \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

The upper bound of the lemma follows. ∎

Remark 2: An in depth analysis of $\mathbf{tr}(P(\mathcal{G},\widetilde{\mathcal{R}}_d))$ was undertaken for the specialized class of graphs trees $\mathcal{T}$ [12]. To apply some of these results to more generalized connected graphs we consider any spanning tree $\mathcal{T}$ of a connected graph $\mathcal{G}$. In terms of our electrical resistance analogy, the resistor network corresponding to $\mathcal{T}$ is formed by removing resistors from the resistor network corresponding to $\mathcal{G}$. From [14], applying Rayleigh's Monotonicity Principle leads to $\mathbf{tr}(P(\mathcal{G},\widetilde{\mathcal{R}}_d)) \leq \mathbf{tr}(P(\mathcal{T},\widetilde{\mathcal{R}}_d))$, i.e., the metrics on a graph are bounded above by the corresponding measures on their respective spanning trees.

We now propose a protocol over the spanning trees of $\mathcal{G}$ with the objective of increasing $\mathbf{tr}\left(P(\mathcal{G},\mathcal{R})\right)$ via increasing $|\mathcal{E}_R|$ and $E_{\text{eff}}(v_i)$ for all $v_i \in \pi(\mathcal{E}_R)$ as illustrated through Lemma 2. The protocol involves edge trades between neighboring agents executed concurrently and/or in a random agent order while maintaining a connected tree at each iteration. The approach is to randomly select a spanning tree $\mathcal{T}$ of $\mathcal{G}$ apply **Protocol 1** for some number of edge trades and then repeat with a new spanning tree. In the following protocol, we denote by $\mathcal{I}(v_i)$ the set of all agents that are neighbors of v_i and lie on the shortest path between v_i and any $r_j \in R$. In this direction, let us first define the special set of agents that lie on any of the shortest paths between agents in $\mathcal{R}$ as *main path agents*, i.e., those agents such that are leader or with $|\mathcal{I}(v_i)| > 1$. The protocol involves two conditions; one to increase the degree of agents in R thus increasing $|\mathcal{E}_R|$, the other to increase the effective resistance for agents in set $\pi(\mathcal{E}_R)$. We have previously presented the following lemmas and we refer to reader to [12] for the corresponding proofs.

Lemma 3: [12] Under both conditions of **Protocol 1**, $|\mathcal{E}_R|$ only increases.

Lemma 4: [12] For a tree $\mathcal{T}$, under the second conditions of **Protocol 1**, $E_{\text{eff}}(v_i)$ for $v_i \in \pi(\mathcal{E}_R)$, monotonically increases.

Lemma 3 involves compressing the network about the main path agents. On the other hand, Lemma 4 involves adding agents to the main path of $\mathcal{T}$ and in doing so elongates the main path.

Protocol 1 Increasing $tr(P(\mathcal{T},\mathcal{R}))$ edge swap

```
foreach Agent v_i do
    if {v_k} = I(v_i), ∃v_j ∈ N(v_i) and v_j ≠ v_k then
        E → E \ {v_i, v_j} ∪ {v_j, v_k}
    end
    if |I(v_i)| > 1, ∃v_j, v_k ∈ N(v_i), v_j ∈ I(v_i) and v_k ∉ I(v_i) then
        E → E \ {v_i, v_j} ∪ {v_j, v_k}
    end
end
```

We now proceed to address the order of the LQG controller, specifically, applying model reduction so as to achieve lower order controllers for the networked UAVs.

IV. Model Reduction

The LQG, and more generally the $\mathcal{H}_2$ design framework, produce controllers of order at least equal to that of the underlying plant, and usually higher because of the inclusions of dynamic weights. The corresponding control laws may be too involved with regards to practical implementation and simpler designs are then sought. For this purpose, one can either reduce the order of the plant model prior to controller design, or reduce the controller in the final stage, or both. We will be examining the case where the plant model is reduced a priori specifically using model truncation but in such a way that the reduced order system corresponds to a leader-follower graph.

Model truncation involves partitioning the state vector x into $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where x_2 is the vector of $n-k$ states which will be removed rearranging the system model such that,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$
$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + Du.$$

A k-th order truncation subsequently produces the approximate model,

$$\dot{x}_1 = A_{11}x_1 + B_1 u$$
$$y = C_1 x_1 + Du,$$

with transfer function $G_a(s) = C_1\left(sI - A_{11}\right)^{-1} B_1 + D$. A common error metric is the error with respect to the Hankel norm and infinity norm between new full model system $G(s)$ and truncated model $G_a(s)$, i.e., $\|G(s) - G_a(s)\|_H$ and $\|G(s) - G_a(s)\|_\infty$, respectively. The Hankel and infinity norm of any stable transfer function $E(s)$ is defined as $\|E(s)\|_H := \sqrt{\max_i |\lambda_i(PP_{obs})|}$ and $\|E(s)\|_\infty := \max_w \bar{\sigma}\left(G(jw)\right)$, where P and P_{obs} are the controllability and observability gramian of $E(s)$ respectively and $\bar{\sigma}(\cdot)$ denotes the singular value of a matrix. The optimal k-th order truncation $G_a^H(s)$ with respect to the Hankel norm is

$$\left\|G(s) - G_a^H(s)\right\|_H = \sigma_{k+1}, \qquad (7)$$

where $\sigma_k = \sqrt{\lambda_k(PP_{obs})}$. Another common form of truncation is balance truncation and balanced realization $G_a^B(s)$ which boasts the bound

$$\left\|G(s) - G_a^B(s)\right\|_\infty \leq 2(\sigma_{k+1} + \cdots + \sigma_n). \quad (8)$$

These methods and their corresponding performance guarantees will be used as benchmarks for our truncation method. For descriptions of these and forms of truncation we refer the reader to [8].

We now proceed with a graph based form of model truncation. A partition of a graph is the grouping of nodes into subsets called cells. The i-th cell is denoted C_i. An *equitable partition* is one in which each node in C_j has the same number of neighbors in C_i for all i, j. A *distance partition* is one in which each cell is composed of all nodes the same distance from some node v_i. We define a distance partition over $\breve{\mathcal{G}} \setminus \mathcal{E}^r$ defined with respect to the leaders, where $\breve{\mathcal{G}} \setminus \mathcal{E}^r$ is the graph $\breve{\mathcal{G}}$ without those edges connected between leaders. If each leader is the sole member of a cell and a cell with more than one member exists and the partition is equitable then it has been shown that the graph is uncontrollable [9] and as such there exists a non-empty uncontrollable subspace which can be truncated from the model without introducing any model error. Further it was shown that symmetry within the graph structure with respect to the leader agents induce an equitable partition [9].

We now have a framework to generate a graph based reduced model by fusing together the states of all agents within a cell. Specifically the reduced model's states are $z_j = \frac{1}{|C_j|}\sum_{v_i \in C_j} x_i$ for all j, i.e., the average state value within the cell. For the case where the partition is an *equitable distance partition* with respect to some r_i, then the same partition will be generated if it was calculated with respect to any r_j. Further as mentioned above, this model reduction has exactly the same input-output characteristics as the full state system, i.e., no model error is introduced. This method for model reduction was introduced for a single leader in [10]. As most leader-follower systems do not have an equitable distance partition, the partition depends on the selection of the leader r_i that defines it. To remove this dependence and provide equal contribution from all leaders we slightly adapt the distance partition and name the adaption a *leader partition*. We define this new partition as follows.

Definition 1: A *leader partition* is composed of cells such that all nodes are initially designated to *leader cells* $\widetilde{C}_j$, where nodes who share the same closest leader are grouped together, i.e., $\widetilde{C}_j = \{v_i : r_j = \text{argmin}_k d(r_k, v_i)\}$. If nodes are equidistant between a set of leaders then membership in one of the corresponding leader cells is randomly chosen. Each leader cell $\widetilde{C}_j$ is then distance partitioned with respect to their corresponding leader node r_j to form leader partition cells.

Our model reduction procedure is to fuse the states of the all agents within each of the leader partition cells. In the case where the leader-follower system contains an equitable distance partition then combinations of the leader partition cells form the distance partition cells. Consequently, the leader partition model reduction would introduce no model errors. The worth of this model reduction process is that it exploits the structural components of the graph when they are close to symmetric, introducing only small model errors corresponding to ignored asymmetries.

The reduced order model has system matrices

$$\begin{aligned} A_r &= STA(\mathcal{G}, \mathcal{R})T^T S \in \mathbb{R}^{n_r \times n_r} \\ B_r &= STB(\mathcal{R}) \in \mathbb{R}^{n_r \times r} \\ C_r &= T^T S \in \mathbb{R}^{n \times n_r}, \end{aligned}$$

where $S \in \mathbb{R}^{n_r \times n_r}$, n_r are the number of cells in the leader partition, $[S]_{ii} = \frac{1}{\sqrt{|C_i|}}$, $T \in \mathbb{R}^{n_r \times n}$ and $[T]_{ij} = 1$ of $v_i \in C_j$ and $[T]_{ij} = 0$ otherwise.

Graphically, $\widetilde{A} = TA(\mathcal{G}, \mathcal{R})T^T$ and $\widetilde{B} = TB(\mathcal{R})$, correspond to the state matrix of the reduced order leader-follower graph $\widetilde{\breve{\mathcal{G}}} = \left(\widetilde{V}, \widetilde{E}\right)$. This can be graphically formed by fusing the nodes of $\breve{\mathcal{G}}$ corresponding to the leader partition and reweighting the edges accordingly. Therefore, each node in $\widetilde{\breve{\mathcal{G}}}$ corresponds to a fused cell and the weight on edge $(v_i, v_j) \in \widetilde{E}$, $\widetilde{w}_{ij}$, is equal to the number of edges between nodes in C_i and in C_j in $\breve{\mathcal{G}}$.

The interpretation of A_r and B_r is a graph with self loops and can be formed by reweighting the edges of $\widetilde{\breve{\mathcal{G}}}$ from $\widetilde{w}_{ij}$ to w_{ij} such that $w_{ij} = \frac{1}{\sqrt{|C_i||C_j|}}\widetilde{w}_{ij}$, for $v_i, v_j \notin R$ and $w_{ij} = \frac{1}{\sqrt{|C_i|}}\widetilde{w}_{ij}$ for $v_j \in R$ otherwise. For the self loops one has,

$$\begin{aligned} w_{ii} &= \frac{1}{|C_i|}\sum_{v_j \neq v_i} \widetilde{w}_{ij} - \sum_{v_j \neq v_i} w_{ij} \\ &= \frac{1}{|C_i|}\sum_{v_j \neq v_i} \widetilde{w}_{ij} - \frac{1}{\sqrt{|C_i|}}\Big(\sum_{v_j \notin R \cup \{v_i\}} \frac{1}{\sqrt{|C_j|}}\widetilde{w}_{ij} + \sum_{v_j \in R} \widetilde{w}_{ij}\Big) \end{aligned}$$

Among the benefits of this type of reduction is a simple graphical approach to model reduction compared to traditionally more numerically intense model reduction, e.g., balanced residualization, as well as the potential to apply graph theoretic analysis on the reduced model.

V. Simulations

The stabilizing compensator in §II was applied to the 40-agent UAV network in Figure 2 with 3 leaders exposed to wind gusts with parameters; $Q = I_{37\times37}$, $R = 10I_{3\times3}$, $V = 0.1I_{37\times37}$, $H = \mathbf{1}$, $W = 1$, $L = 3.49$ and $\sigma = 10$. The performance with respect to the average state and control with a desired hover command ($x = \mathbf{0}$) are compared to the grounded signal control in Figure 5.

Protocol 1 was applied to the spanning tree of the leader-follower graph $\breve{\mathcal{G}}$. It was run over 10 spanning trees of $\breve{\mathcal{G}}$ and involved 231 edge trades. The resultant leader-follower model had a $\mathbf{tr}(P(\mathcal{G}, \mathcal{R})) = 18.3$ compared to the original system with 5.3. The resultant graph and its response to wind gusts are displayed in Figure 3a) and 5, respectively. The optimal performance cost J^* as defined in (5) decreased from 18329 to 586.

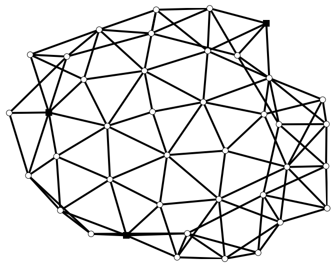

Fig. 2. Agent graph with leader agents (squares).

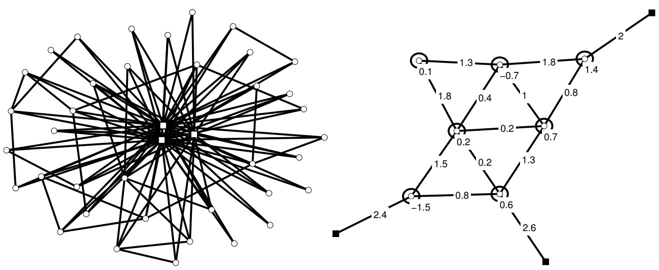

Fig. 3. (a) The resultant leader-follower network after applying **Protocol 1** to 10 spanning trees of the graph in Figure 2. (b) Reduced weighted agent graph corresponding to A_r and B_r with leader agents (squares). Edges from an agent to itself (a self loop) are denoted by a circle around the agent.

The leader partition model reduction was applied to the original network reducing the order of the system from 37 to 7. The model reduced graphs corresponding to A_r, B_r is displayed in Figure 3b). The singular value and impulse response for a selected agent with respect to each of the controls are compared for the full state and reduced state system in Figure 4. Applying the norm metrics we have $\|G(s) - G_a(s)\|_H = 0.45$ and $\|G(s) - G_a(s)\|_\infty = 2.57$ compared to our benchmarks (7) and (8) with $\left\|G(s) - G_a^H(s)\right\|_H = 0.03$ and $\left\|G(s) - G_a^B(s)\right\|_H \leq 0.11$. The reduced model was used to produce an LQG compensator and its performance was compared to the full state compensator in Figure 5.

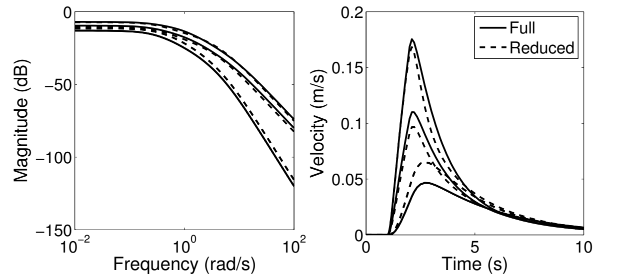

Fig. 4. (a) Singular values and (b) Impulse response for each of the three controller for the 7-th order leader partition model and the full state model (37 state) for a selected agent.

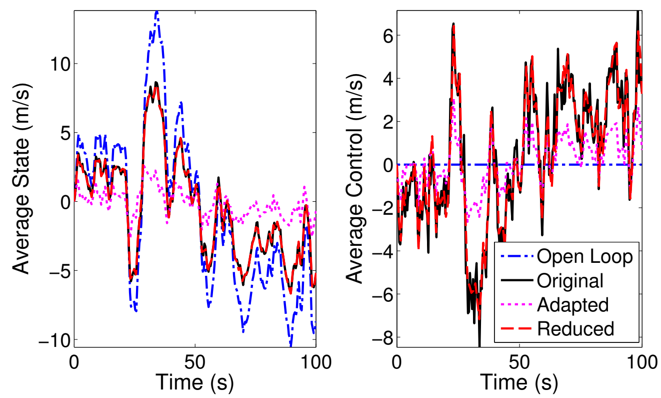

Fig. 5. Average state and control over time for the wind perturbed system corresponding to Figure 2 with; open loop control, original LQG compensator, adapted graph after applying **Protocol 1** and leader partition reduced model.

VI. Conclusion

The main objective of the present work is to propose a network-theoretic approach for the efficient control of leader-follower systems. In particular, the paper first presents an LQG controller formulation for UAV flocking with leaders in the presence of wind gusts. We then proceeded to utilizes the network topology and in particular, adaptively evolve its structure to improve the nominal controller's performance and design. Network properties pertaining to symmetry and agent partitioning provided a network-centric method for model reduction, providing the benefits of decreased controller model order while retaining a graph topology of the reduced system.

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