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Stability Analysis of Nonlinear Networks via M-matrix Theory: Beyond Linear Consensus

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***Abstract*—This paper examines the set of equilibria and asymptotic stability of a large class of dynamical networks with nonidentical nonlinear node dynamics. The interconnection dynamics are defined by M-matrices. An example of such a class of systems include nonlinear consensus protocols as well as other distributed protocols of interest in cooperative control and distributed decision-making. We discuss the model's relationship to the network topology, investigate the properties of its equilibria, and provide conditions for convergence to the set of equilibria. We also provide examples of the versatility of this model in the context of a sensor coverage problem. The model is extended to incorporate additional nonlinearities; an application for this latter model is also provided in the realm of neural networks.**

***Index Terms*—Nonlinear Dynamics Networks; Nonlinear consensus protocol; M-matrices**

I. Introduction

Complex dynamic networks are an integral part of the technological world around us including the internet, power grids, and communication networks, as well as in the world at large such as biological and chemical systems and social networks. An explosion of research in the area of network systems has eventuated [1], [2], [3].

Stability is an important requirement for networked systems, as it is for dynamic systems in general. Examining stability for nonlinear non-networked systems is generally a challenging task with Lyapunov stability theory used as a common tool. A further challenge is that the connection of stable, isolated dynamics does not guarantee network stability with node interactions able to introduce instabilities. The consequence of these challenges is that proving stability of networked systems often involves searching for Lyapunov candidate by trial-and-error. In this paper, we provide a class of nonlinear networks with interconnection dynamics defined by M-matrices that exhibit asymptotic stability.

M-matrices are used to model synchronization in networks [4], population migration [5], Markov processes, and supply and demand in economic systems [6]. M-matrices also appear in the discretization of differential operators. An example is the discretization of diffusion [4] and advection [7] dynamics which generate the in-degree and out-degree Laplacian matrix respectively, and both of which are M-matrices. The in-degree Laplacian matrix forms the basis of consensus models which are effective for distributed information-sharing and control of networked, multi-agent systems in settings such as multi-vehicle control, formation control, swarming, and distributed estimation; see for example [8], [9].

The novelty of our work in the field of nonlinear networks lies in its generality. Nonlinear dynamics over networks has been a largely unexplored research area; [10] provides a good summary of the field. Many nonlinear network results are quite conservative such as assuming uniform node dynamics, undirected network structure and restrictive classes of nonlinear dynamics. Our results are over non-identical nodes, strongly connected directed network topologies, and have mild assumptions on the nonlinear dynamics. We believe that these generalizations are of special interest to the area of nonlinear consensus models. Not surprisingly, there is a great focus in nonlinear network research on globally asymptotically stable equilibrium. However, in many applications, asymptotic convergence to a set of equilibria is of interest, such as Laplacian dynamics [8]. It is with this in mind that we examine this more general type of convergence.

Araki and Kondo [11] decomposed networks into subnetworks obtaining Lyapunov functions for these subsystems, and under DC gain conditions, showed asymptotic stability of the nonlinear system. In [10], the authors provide sufficient conditions for asymptotic convergence over a large class of nonlinear systems but the application of the paper's results does require the construction of Lyapunov functions. Xiang and Chen [12] provide a passivity measure indicating the amount of effort required to stabilize a node. The measure is used to form a network stability condition that guarantees asymptotic stability. Siljak [13] examined convergence of nonlinear network dynamics to a single equilibrium, irrespective of network connectivity.

Recently, there has been interest in nonlinear consensus, a subclass of nonlinear dynamics over networks. Cortes [14] proposed nonlinear distributed algorithms over smooth functions that achieve consensus, with many results assuming balanced graphs. In [15], the authors examined convergence to a set of equilibria, referred to as semistability, for nonlinear consensus over balanced graphs. Convergence of nonlinear consensus algorithms to a single point was examined in [16] using a contracting property. Yu *et al.* [17] examined linear consensus with an additional nonlinear term solely dependent on the agent's state and identical over all agents. We examine a similar model, in the extension component of this paper, but with nonlinear consensus and the addition of non-identical nonlinear terms.

The organization of the paper is as follows. We begin by defining the class of nonlinear network models and related

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background. We provide graph related features of the network dynamics. The left and right null-spaces of M-matrices are used to characterize the equilibria of the model as well as an invariance property of the system. An asymptotic convergence result is then presented and justified by exploiting properties of M-matrices. An example of the network model is investigated, a sensor network problem based on the model in [7]. Finally, the model is extended to incorporate individual node dynamics independent of the network structure. An illuminating neural network example [11] is provided which demonstrates the utility of this extended model.

II. Background and Model

We provide a brief background on constructs and models that will be used in this paper.

We define the vectors $\mathbf{1} := [1, 1, \ldots, 1]^T$ and $\mathbf{0} := [0, 0, \ldots, 0]^T$. For column vector $v \in \mathbb{R}^p$, v_i or $[v]_i$ denotes the ith element. For matrix $M \in \mathbb{R}^{p \times q}$, $[M]_{ij}$ denotes the element in its ith row and jth column. We write $M \succ 0$ ($M \succeq 0$) if M is a positive definite (semidefinite) matrix.

We consider a multi-agent network of n coupled nodes with each nodes' state $x_i(t) \in \mathbb{R}$ at time t. The system is described by the differential equations

$$\dot{x}_i(t) = -a_{ii} f_i\left(x_i(t)\right) + \sum_{j \neq i} a_{ij} f_j\left(x_j(t)\right), \; i = 1, \ldots, n,$$

where $f_i : \mathbb{R} \to \mathbb{R}$ is a scalar-valued function representing the ith node dynamics, and $a_{ij} \in \mathbb{R}$ is nonnegative for all $i, j \in \{1, \ldots, n\}$. In a more compact form, the dynamics can be written as

$$\dot{x} = -Af(x), \tag{1}$$

where $x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$, $f(x) = [f_1(x_1), \ldots, f_n(x_n)]^T \in \mathbb{R}^n$ and the matrix A is defined as

$$A = \begin{bmatrix} a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & a_{22} & -a_{23} & \vdots \\ \vdots & & \ddots & -a_{n-1,n} \\ -a_{n1} & \cdots & -a_{n,n-1} & a_{nn} \end{bmatrix}.$$

In this way, the A matrix codifies the nodes' interconnections. We define the class of matrices of this form as $Z_n = \{A = (a_{ij}) \in \mathbb{R}^{n \times n} : a_{ii} \geq 0, a_{ij} \leq 0, i \neq j\}$. We make the assumption that $f(x)$ is continuous and $\int_0^{x_i} f_i(y)\, dy$ is radially unbounded, i.e., ($\int_0^{x_i} f_i(y)\, dy \to \infty$ as $|x_i| \to \infty$), for $i = 1, \ldots, n$. The class of functions that satisfies this requirement is denoted by F_0. Further, we assume that A in model (1) is a member of the subclass of matrices known as irreducible M-matrices denoted by M_0. These terms are defined as follows:

Definition 1. [6] A matrix $A \in Z_n$ is called a nonsingular (singular) M-matrix if the real part of every eigenvalue of A is positive (nonnegative).

Definition 2. [18] A matrix $A \in \mathbb{R}^{n \times n}$ is reducible if and only if, for some permutation matrix P, the matrix $P^T AP$ has triangular block form. A matrix that is not reducible is irreducible.

M-matrices appear in a myriad of problems including biological, physical, and social sciences; for examples see [19], [20]. One of the famous matrices is the in-degree Laplacian matrix used in consensus (or synchronization) dynamics. An attraction of the Laplacian-based dynamics is the structure of the networks can be coded into the in-degree Laplacian matrix, which will be defined presently, and consequently graph-theoretic results can be applied to the dynamic models. As for the in-degree Laplacian matrix, the underlying coupling network between agents in the network are encoded in the M-matrix A.

The network topology of model (1) can be formed using a graph realization of A. We define a weighted digraph $\mathcal{G} = (V, E, W)$ which is characterized by a node set V with cardinality n, an edge set E comprised of pairs of nodes with cardinality m, and a weight set W with cardinality m, where information flows from node i to j if $(i, j) \in E$ with edge weight $w_{ji} \in W$. The corresponding graph realization of matrix A is a digraph $\mathcal{G}$ where $(i, j) \in E$ if $a_{ji} > 0$ and $i \neq j$ with corresponding weight $a_{ji} = w_{ji} \in W$. Some special classes of digraphs are undirected graphs where $(i, j) \in E$ implies $(j, i) \in E$, balanced graphs where $\sum_{j \neq i} w_{ij} = \sum_{j \neq i} w_{ji}$ for $i = 1, \ldots, n$, and rooted directed tree graphs where $m = n - 1$ and there exists a node i such that there is a directed path from node i to every other node in the graph. Because of the link between the off-diagonals of matrix A and its digraph realization $\mathcal{G}$, we will often denote the matrix A as $A(\mathcal{G})$ to emphasize the underlying graph.

As the diagonal of the M-matrix is ignored in the graph realization there is more than one M-matrix for any given digraph. Consequently, when provided with the digraph $\mathcal{G}$ and the diagonal of $A(\mathcal{G})$ the M-matrix $A(\mathcal{G})$ is fully defined. The in-degree Laplacian matrix $L_{in}(\mathcal{G})$ and out-degree Laplacian matrix $L_{out}(\mathcal{G})$ have the same graph realization $\mathcal{G}$ but differ in their diagonals defined with $[L_{in}(\mathcal{G})]_{ii} = \sum_{j \neq i} a_{ij}$ and $[L_{out}(\mathcal{G})]_{ii} = \sum_{j \neq i} a_{ji}$ for all $i = 1, \ldots, n$. These M-matrices exhibit the special conditions $L_{in}(\mathcal{G})\mathbf{1} = \mathbf{0}$ and $\mathbf{1}^T L_{out}(\mathcal{G}) = \mathbf{0}^T$.

The related digraph $\mathcal{G}$ of an M-matrix $A(\mathcal{G})$ can be used to extract matrix properties of $A(\mathcal{G})$, even though there is not a one-to-one correspondence between $A(\mathcal{G})$ and $\mathcal{G}$. If the digraph $\mathcal{G}$ is undirected then $A(\mathcal{G})$ is symmetric. The irreducibility of a matrix A can be established using the digraph realization of A. This result is summarized in the following proposition.

Proposition 1. *[20] A matrix $A(\mathcal{G})$ is irreducible if and only if $\mathcal{G}$, the digraph realization of A, is strongly connected. A graph is strongly connected if between every pair of distinct nodes there exists a directed path.*

One of the attractions of irreducible M-matrices is that they fit into a special class of diagonally semi-stable matrices that will be exploited in the core result of this paper (Lemma 5). This property is formally stated in the following.

Proposition 2. *[6] If A is a irreducible M-matrix then there* *exists a positive diagonal matrix $D \in \mathbb{R}^{n\times n}$ such that*

$$DA + A^T D \succcurlyeq 0.$$

Further, if A is nonsingular $DA + A^T D \succ 0$.

We will not include a repetition of the proof of Proposition 2 by [6] in this paper but we will re-derive this proposition for the special case where the M-matrix is the in-degree Laplacian matrix. We will deviate from the traditional derivation and use a graph-based proof to illustrate the utility of the digraph realization of M-matrices.

Proposition 3. *For a strongly connected graph $\mathcal{G}$ and $v^T L_{in}(\mathcal{G}) = 0$, $DL_{in}(\mathcal{G}) = L_{in}(\widetilde{\mathcal{G}})$ for some balanced graph $\widetilde{\mathcal{G}}$ where D is a positive diagonal matrix with $[D]_{ii} = v_i$ for $i = 1, \dots, n$ and $DL_{in}(\mathcal{G}) + L_{in}(\mathcal{G})^T D \succeq 0$.*

Proof: It has previously been shown in [21] that the eigenvector $v = [v_1, v_2, \dots, v_n]^T$ can be found by examining the digraph $\mathcal{G} = (V, E, W)$ specifically

$$v_i = \sum_{T\in\mathcal{T}_i} \prod_{e_{kj}\in T} w_{jk},\ i = 1, \dots, n$$

where $\mathcal{T}_i$ is the family of rooted directed spanning trees over $\mathcal{G}$ with node i as the root, and e_{ij} is an edge in a given tree T with weight w_{ji}. As $\mathcal{G}$ is strongly connected, $\mathcal{T}_i$ is nonempty and so $v_i > 0$, for all i.

Consider a new graph formed from the old with $\widetilde{\mathcal{G}} = \left(V, E, \widetilde{W}\right)$ where $\widetilde{W} = \{\widetilde{w}_{jk}\}$ and $\widetilde{w}_{jk} = v_j w_{jk}$ then $L_{in}(\widetilde{\mathcal{G}}) = DL_{in}(\mathcal{G})$. Further, for the left eigenvector $\tilde{v}$ corresponding to the zero eigenvalue has

$$\begin{aligned}
\widetilde{v}_i &= \sum_{T\in\mathcal{T}_i} \prod_{e_{kj}\in T} \widetilde{w}_{jk} \\
&= \sum_{T\in\mathcal{T}_i} \prod_{e_{kj}\in T} v_j w_{jk} \\
&= \left(\sum_{T\in\mathcal{T}_i} \prod_{e_{kj}\in T} w_{jk}\right) \prod_{j\neq i} v_j \\
&= v_i \prod_{j\neq i} v_j \\
&= \prod_{i=1}^{n} v_j.
\end{aligned}$$

Here, $\widetilde{v}_i = \widetilde{v}_j$ for all $i, j \in \{1, \dots, n\}$, thus $\widetilde{v} = \left(\prod_{i=1}^n v_j\right) \mathbf{1}$, or normalized as $\widetilde{v} = \frac{1}{n}\mathbf{1}$. Therefore, as the left eigenvector corresponding to the zero eigenvalue is $\mathbf{1}$ and similarly the right eigenvector is $\mathbf{1}$ (as $L_{in}(\mathcal{G})$ is an in-degree Laplacian), then $\widetilde{\mathcal{G}}$ is a balanced graph [4]. A property of balanced graphs is that $L_{in}(\widetilde{\mathcal{G}}) + L_{in}(\widetilde{\mathcal{G}})^T \succeq 0$, hence $DL_{in}(\mathcal{G}) + L_{in}(\mathcal{G})^T D \succeq 0$. ■

A corollary of Proposition 3 pertains to forming balanced digraphs:

Corollary 3. *The edges of a strongly connected digraph can always be re-weighted to achieve a balanced connected digraph.*

The right and left null spaces of an irreducible M-matrix play an important role in the dynamics of model (1) in particular in relation to the model's set of equilibria. The following proposition will be exploited shortly:

Proposition 4. *[6] For $A \in M_0$, either A is invertible or, the rank of A is $n-1$, in which case every element in the right and left eigenvector corresponding to the zero eigenvalue are nonzero, and share the same sign.*

We will now investigate the set of equilibria and an invariant property of model (1), present the main convergence proof of the paper, and provide examples.

III. Set of Equilibria and Convergence

Model (1), under the assumptions $f(\cdot) \in F_0$ and $A \in M_0$, has in general many equilibria which we define by the set

$$\mathcal{A} = \{x \in \mathbb{R}^n | Af(x) = 0\}.$$

Consequently, the set $\mathcal{A}$ is completely defined by the properties of A and $f(\cdot)$. If $\mathcal{A}$ is composed of isolated equilibria and $x(t) \to \mathcal{A}$ then $x(t)$ will converge to some $x_e \in \mathcal{A}$.

From Proposition 4, A is nonsingular, or singular with rank $n-1$, we will proceed to investigate $\mathcal{A}$ for each of these cases.

For the case where A is nonsingular then this set becomes $\mathcal{A} = \{x \in \mathbb{R}^n | f(x) = 0\}$. As $f(x)$ is continuous, this implies that an equilibrium $x_e \in \mathcal{A}$ is isolated if and only if, for some ball around $[x_e]_i$, $f_i(x_i) = 0$ implies that $x_i = [x_e]_i$. If $f(x)$ is differentiable at x_e then if $\frac{d}{dx_i} f_i([x_e]_i) \neq 0$ for $i = 1, \dots, n$, there exists the aforementioned ball around x_i for $i = 1, \dots, n$,. Therefore the equilibrium is isolated.

When A is singular there exists a right and left eigenvector corresponding to zero, v and w respectively, such that $w^T A = \mathbf{0}^T$ and $Av = \mathbf{0}$. Using this fact, $\frac{d}{dt}\left\{w^T x(t)\right\} = -w^T Af(x) = 0$ and the quantity $w^T x(t)$ remains invariant under model (1). For special matrices like the out-degree Laplacian matrix $L_{out}(\mathcal{G})$, this is a familiar property with $\mathbf{1}^T L_{out}(\mathcal{G}) = 0$, the conserved quantity is $\sum_{i=1}^n x_i(t)$.

The eigenvector v provides an alternate definition of the set of equilibria, specifically

$$\mathcal{A} = \{x \in \mathbb{R}^n | f(x) = \beta v, \beta \in \mathbb{R}\}. \tag{2}$$

Proposition 5. *The equilibrium $x_e \in \mathcal{A}$, corresponding to a singular A, is isolated if $f(x)$ is differentiable at x_e, $\frac{d}{dx_i} f_i([x_e]_i) \neq 0$ for all $i = 1, \dots, n$, and*

$$\sum_{i=1}^{n} \frac{v_i w_i}{\frac{d}{dx_i} f_i([x_e]_i)} \neq 0,$$

where v and w are the right and left eigenvectors of A corresponding to the zero eigenvalue.

Proof: Consider the function

$$g(x, \beta) = [g_1(x_1, \beta), \dots, g_n(x_n, \beta), g_{n+1}(x)]^T,$$

$$g_i(x_i, \beta) = f_i(x_i) - v_i\beta,\ i = 1, \dots, n$$

and

$$g_{n+1}(x) = w^T(x - x_0),$$

where $\beta \in \mathbb{R}$, and x_0 is the initial condition. If x_e is an equilibrium, then for some β_e, $g(x_e, \beta_e) = 0$.

As $f(x)$ is differentiable at x_e then $g(x_e, \beta_e)$ is differentiable at (x_e, β_e). Calculating the Jacobian of $g(\cdot)$ about (x_e, β_e),

$$\nabla g(x, \beta) = \begin{bmatrix} \frac{d}{dx_1} g_1(x_1, \beta) & 0 & 0 & \frac{d}{d\beta} g_1(x_1, \beta) \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & \frac{d}{dx_n} g_n(x_n, \beta) & \frac{d}{d\beta} g_n(x_n, \beta) \\ \frac{d}{dx_1} g_{n+1}(x) & \cdots & \frac{d}{dx_n} g_{n+1}(x) & 0 \end{bmatrix} = \begin{bmatrix} \frac{d}{dx_1} f_1(x_1) & 0 & 0 & v_1 \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & \frac{d}{dx_n} f_n(x_n) & v_n \\ w_1 & \cdots & w_n & 0 \end{bmatrix}.$$

The determinant of the Jacobian is

$$\det(\nabla g(x, \beta)) = -\sum_{i=1}^{n} v_i w_i \left(\prod_{j \neq i} \frac{d}{dx_j} f_j(x_j) \right).$$

From the condition $\frac{d}{dx_i} f_i([x_e]_i) \neq 0$ for all $i = 1, \ldots, n$,

$$\det(\nabla g(x_e, \beta_e)) = -\left(\prod_{i=1}^{n} \frac{d}{dx_i} f_i([x_e]_i) \right) \sum_{i=1}^{n} \frac{v_i w_i}{\frac{d}{dx_i} f_i([x_e]_i)}.$$

Consequently, if $\sum_{i=1}^{n} \frac{v_i w_i}{\frac{d}{dx_i} f_i([x_e]_i)} \neq 0$ then $\det(\nabla g(x_e, \beta_e)) \neq 0$. By the Inverse Mapping Theorem [22], when $\det(\nabla g(x_e, \beta_e)) \neq 0$ there exists a ball around (x_e, β_e) such that $g(x, \beta)$ is one-to-one. In other words, there exists a region about (x_e, β_e) such that there are no other points satisfying $g(x, \beta) = 0$, i.e., no other equilibrium points.

Hence, x_e is an isolated equilibrium. ■

From Proposition 5 and the fact that $v_i w_i$ is nonzero and has the same sign for all $i = 1, \ldots, n$ (Proposition 4), we have the following:

Corollary 4. *The equilibrium $x_e \in \mathcal{A}$, corresponding to a singular A, is isolated if $f(x)$ is differentiable at x_e and $\frac{d}{dx_i} f_i([x_e]_i) > 0$ for all $i = 1, \ldots, n$.*

We now provide the main result of the paper which describes the asymptotic convergence of the model.

Lemma 5. *Under the assumption that $f_i(x) \in F_0$ and $A \in M_0$, $x(t) \to \mathcal{A}$ for all initial conditions in model (1). If $\mathcal{A}$ is composed of a finite number of isolated equilibria then $x(t) \to x_e$ for some $x_e \in \mathcal{A}$.*

Proof: As $A \in M_0$ then from Proposition 2 there exists a positive diagonal matrix D with $DA + A^T D \succeq 0$. Consider the Lyapunov function $V(x) : \mathbb{R}^n \to \mathbb{R}$ defined by

$$V(x) = \sum_{i=1}^{n} [D]_{ii} \int_0^{x_i} f_i(y)\, dy,$$

which is continuous, differentiable and radially unbounded. Taking the derivative of $V(x)$,

$$\begin{aligned} \dot{V}(x) &= \sum_{i=1}^{n} [D]_{ii} f_i(x_i) \dot{x}_i \\ &= f(x)^T D \dot{x} \\ &= -f(x)^T D A f(x) \\ &= -\frac{1}{2} f(x)^T D A f(x) - \frac{1}{2} f(x)^T A^T D f(x) \\ &= -\frac{1}{2} f(x)^T \left(DA + A^T D \right) f(x). \end{aligned}$$

Now, from the property $DA + A^T D \succeq 0$ then $\dot{V}(x) \leq 0$. Defining $E = \left\{ x \in \mathbb{R}^n | \dot{V}(x) = 0 \right\}$, then $\mathcal{A} \subseteq E$ and $\mathcal{A}$ is the largest invariant set of $\mathbb{R}^n$. Since $V(x)$ is radially unbounded and $\dot{V}(x) \leq 0$ for all x, the set $\Omega = (x \in \mathbb{R}^n | V(x) \leq c)$ is a compact, positively invariant set. From LaSalle's theorem [23], we conclude that every trajectory starting in Ω asymptotically converges to the set $\mathcal{A}$ as $t \to \infty$. Moreover since $V(x)$ is radially unbounded, the conclusion is global, as for every $x(0)$ there exists a c such that $x(0) \in \Omega$. Further, if all equilibria are isolated then the trajectory must approach one of these equilibria. ■

Next we showcase an example of networked, multi-agent systems which fit under the umbrella of model (1). We focus on a sensor coverage problem.

A. Example 1 - Sensor Network

We consider a sensor surveillance task, operating in a land corridor of length d_w, where the sensors are directional with a narrow cone of observation. A set of n sensors are randomly placed pointing east in the corridor, which is oriented west to east. The objective is to acquire good coverage of the corridor while satisfying a total sensor power constraint by trading power between neighboring sensors. Let $x_i(t) \geq 0$ be the fraction of total network power and $z_i(t) \geq 0$ be the coverage of sensor i at time t. We utilize a coverage model of the form $z_i(t) = \beta \sqrt{x_i(t)}$ (wedge-shaped sensor) for some $\beta > 0$. We subsequently require that the total power of the sensor network is constant, i.e., $\sum x_i(t) = 1$ for all t, while maintaining good coverage. We refer the reader to [7] for a discussion of the measure of "good coverage", which aims to minimize gaps between sensor observation areas along the east-west axis and state the derived requirement as

$$\tilde{z}_i = \frac{1}{\delta_i} \sum_{(k,i) \in E} \tilde{z}_k, \text{ for } i = 1, \ldots, n, \tag{3}$$

where $\delta_i > 0$ in the number of in-degree neighbors of i and E is the edges of some graph $\mathcal{G}$. The graph $\mathcal{G}$ is defined in the following passage.

Let the position of sensor i along the east-west axis be p_i. We assume, since the sensors are in a land corridor, that the difference in sensor positions in the north-south axis is small, and subsequently, if the communication range on sensors is $d_c > 0$, $d_c \in \mathbb{R}$, that sensor i is in communication range of sensor j if $|p_i - p_j| \leq d_c$. Additionally, we assume that if sensors are close to the east or west end of the corridor, communications can be relayed between them via infrastructure at the ends of the corridor. A communication graph $\mathcal{G} = (V, E, W)$ is defined such that if sensor j is within communication range of sensor i and $p_j > p_i$ then $(i, j) \in E$ with the exception of the sensors at the ends of the corridor where if $p_j - p_i \geq d_w - d_c$ then $(j, i) \in E$. The graph is unweighted with $w_{ij} = 1$ for all $(j, i) \in E$.

Assuming d_c is large enough to form $\mathcal{G}$ which is strongly connected. The selected dynamic model is the out-degree Laplacian dynamics (or advection dynamics)

$$\dot{x} = -L_{out}(\mathcal{G}) f(x), \tag{4}$$

where $f(x) = [f_1(x_1), \ldots, f_n(x_n)]^T$, $f_i(x_i) = \sqrt{x_i}$ for $x_i \geq 0$ and $f_i(x_i) = 0$ otherwise. The choice of this dynamics model is justified by the dynamics' equilibrium satisfying

$$\sqrt{x_i} = \frac{1}{\delta_i} \sum_{(k,i) \in E} \sqrt{x_k}, \text{ for } i = 1, \ldots, n, \tag{5}$$

which is equivalent to (3). From Proposition 4 and the fact $\mathbf{1}^T L_{out}(\mathcal{G}) = \mathbf{0}^T$, then $\sum x_i(0) = \mathbf{1}^T x(0)$. There is exactly one equilibrium satisfying condition (5) when $\mathbf{1}^T x(0) > 0$, specifically x_e where $[x_e]_i = \frac{1}{n} \mathbf{1}^T x(0) v_i^2$ and v is the normalized right eigenvector of $L_{out}(\mathcal{G})$ corresponding to the zero eigenvalue. As $f(x) \in F_0$ and $L_{out}(\mathcal{G}) \in M_0$ then by Lemma 5, the dynamics (4) will converge to x_e.

We assume that all sensors are initialized with a feasible power, i.e., $\sum_{i=1}^n x_i(0) = 1$ and $x_i(0) \geq 0$ for $i = 1, \ldots, n$. We apply this approach to a $d_w = 40$ m long land corridor containing 40 randomly placed sensors. The initial power fraction was assigned randomly and $d_c = 1.75$ m dictates the topology of the graph. The final equilibrium power x_e overlayed on the graph $\mathcal{G}$ is displayed in Figure 1. Figure 2 depicts the observation cones and uncovered and redundantly covered sensor areas for a) the optimal power usage from all sensors (providing sensor coverage sufficient to cover the east-west axis without redundant coverage), b) the equilibrium power usage obtained using the advection protocol, and c) a uniform power usage for all sensors. We find that the minimum power requirement by the advection equilibrium power to cover the corridor is within 1.25 times of the optimal power.

We now extend model (1) to incorporate additional terms pertaining to the individual node's dynamics separate from the network dynamics.

IV. Extension

Consider a modification of model (1) via the addition of a nonlinear term to each agent that is only dependent on that agent's state. This term represents the node's dynamics independent of network interactions. The model then becomes

$$\dot{x} = -g(x) - Af(x), \tag{6}$$

where $g(x) = [g_1(x_1), \ldots, g_n(x_n)]^T \in \mathbb{R}^n$ and $g_i : \mathbb{R} \to \mathbb{R}$ is a scalar-valued function. We assume that $f_i(x_i) g_i(x_i) \geq 0$.

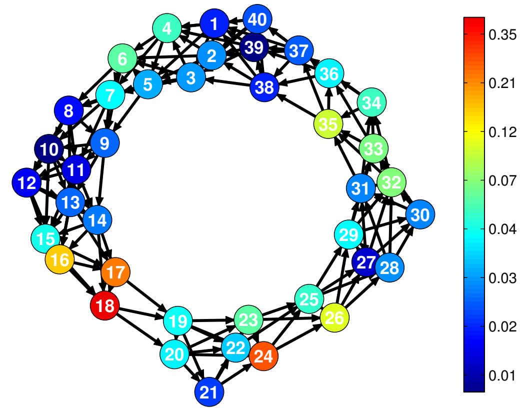

Fig. 1. Sensor graph with color gradations corresponding to the equilibrium power x_e. Nodes are numbered from west to east.

The class of functions that satisfy this requirement we denote by $g(\cdot) \in G_0(f(\cdot))$. The equilibria set for model (1) is

$$\mathcal{B} = \{x \in \mathbb{R}^n | g(x) + Af(x) = 0\}.$$

We now provide an equivalent asymptotic convergence property to Lemma 5.

Lemma 6. *Under the assumption that $f(x) \in F_0$, $g(x) \in G_0(f(\cdot))$ and $A \in M_0$, $x(t) \to \mathcal{B}$ for all initial conditions in model* (1)*. If $\mathcal{B}$ is composed of a finite number of isolated equilibria then $x(t) \to x_e$ for some $x_e \in \mathcal{B}$.*

Proof: Using the same Lyapunov function $V(x)$ as in the proof of Lemma 5 and taking the derivative of $V(x)$ we have

$$\begin{aligned} &\dot{V}(x) \\ &= \sum_{i=1}^{n} [D]_{ii} f_i(x_i) \dot{x}_i \\ &= -\frac{1}{2} f(x)^T \left(DA + A^T D\right) f(x) - f(x)^T D g(x) \\ &= -\frac{1}{2} f(x)^T \left(DA + A^T D\right) f(x) - \sum_{i=1}^{n} [D]_{ii} f_i(x_i) g_i(x_i). \end{aligned}$$

Now, from the property $DA + A^T D \succeq 0$ and $f_i(x_i) g_i(x_i) \geq 0$ which implies $\sum_{i=1}^n [D]_{ii} f_i(x_i) g_i(x_i) \geq 0$, $\dot{V}(x) \leq 0$ where $\dot{V}(x) = 0$ for the set $\mathcal{B}$. As in the proof for Lemma 5, the set $\Omega = (x \in \mathbb{R}^n | V(x) \leq c)$ is compact and positively invariant. From LaSalle's theorem [23], we conclude that every trajectory starting in Ω asymptotically converges to the set $\mathcal{B}$ as $t \to \infty$. Moreover since $V(x)$ is radially unbounded, the conclusion is global, as for every $x(0)$ there exists a c such that $x(0) \in \Omega$. Further, if all equilibria are isolated then the trajectory must approaches one of these equilibria. ∎

The additive neural network is an example of the model relevant to Lemma 6 and will be described in the following section.

A. Example 2 - Neural Network

We consider a neural network described by the model

$$\dot{x} = -Bx - Af(x), \tag{7}$$

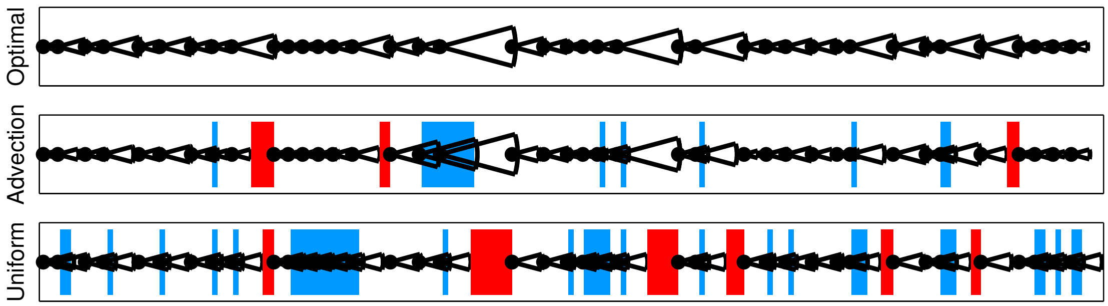

Fig. 2. Optimal, advection dynamics and uniform sensor coverage of the land corridor. The dark (red) shaded bands indicate areas not covered by any sensors, and light (blue) shaded bands indicate areas redundantly covered by multiple sensors.

where $B \in \mathbb{R}^{n\times n}$ is a positive diagonal matrix, $A \in M_0$, $f(x)$ is a differential function with $f_i(0) = 0$, $0 < \frac{d}{dx_i} f_i(x_i) \le \beta$. Consequently, as $f(x)$ is an increasing function $f(x) \in F_0$. If we let $g(x) = Bx$ where $g_i(x_i) = [B]_{ii} x_i$ then as $x_i \ge 0$ ($x_i \le 0$) implies $f_i(x_i) \ge 0$ and $g_i(x_i) \ge 0$ ($f_i(x_i) \le 0$ and $g_i(x_i) \ge 0$), then $g_i(x_i) f(x_i) \ge 0$. Hence, $g(x) \in G_0(f(\cdot))$. Model (7) corresponds to an additive neural network [24]. Here, $x_i(t)$ is the state of a neuron, $-[B]_{ii} x_i$ is a dampening term, $f_i(x_i)$ is its output state which effects the dynamics of other neurons in the network via the weighting matrix A. There is a great interest establishing the global asymptotic stability of such systems with application to optimization and cognitive problems [24], [25], [26].

It was shown in [24] that $x = \mathbf{0}$ is a unique equilibrium. As model (7) satisfies the assumptions of Lemma 6, then $x = \mathbf{0}$ is globally asymptotically stable.

V. Conclusion

This paper presents an analysis of a class of nonlinear dynamic networks involving M-matrices, of which nonlinear consensus is a member. Properties of the model's M-matrix were related to its underlying network interaction topology. We explored the equilibria set of the network and derived sufficient conditions for an isolated equilibrium. Asymptotic stability of the model with non-identical nodes and strongly directed network topology was proven under continuity and boundedness assumption on the nonlinear dynamics. The model was also extended with an additive nonlinear term and a similar convergence condition was provided. Two applications that are examples of M-matrix models were presented. Future work of particular interest involves the introduction of control terms into the dynamics and the subsequent examination of the model's stability.

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