2010 IEEE International Conference on Robotics and Automation
Anchorage Convention District
May 3-8, 2010, Anchorage, Alaska, USA

Semi-Autonomous Networks: Theory and Decentralized Protocols

Airlie Chapman, Eric Schoof, and Mehran Mesbahi

***Abstract*—This paper examines the system dynamics of a networked multi-agent system, operating with a consensus-type algorithm, that can be influenced by external agents. We refer to this class of networks as *semi-autonomous*. We introduce a control scheme for such semi-autonomous networks, involving excitation of the network by the external agents, with the objective of manipulating or steering the network. In this context, we consider the situation where the external agents deliver a constant mean control signal. We proceed to examine the resultant mean and covariance of the network output and relate these quantities to circuit-theoretic notions of the network, quantifying the network's amenability to external signals. Four protocols for tree graphs, to promote and deter convergence and to increase and reduce average variance within the network are then presented. These protocols involve decentralized local edge swaps that can be performed in parallel and asynchronously.**

***Index Terms*—Semi-autonomous networks; Consensus protocol; Graph theory; Coordinated control over networks**

I. Introduction

Consensus-type algorithms provide effective means for distributed information-sharing and control for networked, multi-agent systems in settings such as multi-vehicle control, formation control, swarming, and distributed estimation; see for example, [1], [2], [3], [4]. An appeal of consensus algorithms is their ability to operate *autonomously* over simple trusting agents. This has the added benefit that external (control) agents, perceived as simple agents, can seamlessly attach to the network. These additional agents, ignoring consensus rules, will influence the system dynamics compared to the *unforced* networked system resulting in scenarios such as leader-follower [2] and drift correction [5]. The detriment is that this same approach can be adopted by malicious infiltrating agents. We refer to this class of systems as *semi-autonomous* networks. In such a setting, we examine the effectiveness of a *constant mean control,* in which external (control) agents deliver a signal with a constant mean to neighboring *native* agents in the network. An electrical network analogy is used to measure the average convergence cost of the constant mean control and, via the controllability gramian, its impact on the output variance and output energy. Four decentralized protocols for tree graphs are introduced to perform local edge swaps with the objective of varying the convergence cost and variance of the *forced* system.

The research was supported by AFOSR grant FA9550-09-1-0091 and NSF grant CMMI-0856737. The authors are with the Dept. of Aeronautics and Astronautics, University of Washington, WA 98105. Emails: {airliec, eschoof, mesbahi}@uw.edu.

These approaches provide insight into the *manageability* (friendly control agents) of the underlying coordination network and algorithm. The current work is also part of a more general effort that aims to identify fundamental bounds on the *security* (unfriendly control agents) of coordination algorithms for dynamic systems when controlled or infiltrated by an adversary. As such, our work is related to other research works such as those in computer network security [6], spread of epidemics [7], and predator/prey swarming [8]. The paper complements work on infiltration-adaption and infiltration-detection such as [9], [10].

II. Background and Model

We provide background on constructs and models that will be used in this paper, including abbreviated descriptions on graphs and the consensus protocol, in its unforced and forced versions.

An undirected graph $\mathcal{G} = (V, E)$ is defined by a node set V with cardinality n and an edge set E comprised of a pairs of nodes, where nodes v_i and v_j are adjacent if $\{v_i, v_j\} \in E \subseteq [V]^2$.[1] We denote the set of nodes adjacent to v_i as $\mathcal{N}(v_i)$ and the minimum path length, induced by the graph, between nodes v_i and v_j as $d(v_i, v_j)$. The degree δ_i of node v_i is the number of its adjacent nodes. The degree matrix $\Delta(\mathcal{G})$ is a diagonal matrix with δ_i at position (i, i). The adjacency matrix is a symmetric matrix with $[\mathcal{A}(\mathcal{G})]_{ij} = 1$ when $\{v_i, v_j\} \in E$ and $[\mathcal{A}(\mathcal{G})]_{ij} = 0$ otherwise. The combinatorial Laplacian is defined as $L(\mathcal{G}) = \Delta(\mathcal{G}) - \mathcal{A}(\mathcal{G})$ which is a (symmetric) positive semi-definite matrix. The analysis of this paper will be concerned with the spectrum of the graph Laplacian. That spectrum is assumed to be ordered as $0 = \lambda_1(\mathcal{G}) \leq \lambda_2(\mathcal{G}) \leq \cdots \leq \lambda_n(\mathcal{G})$, where, for brevity, we have used $\lambda_i(\mathcal{G})$ instead of $\lambda_i(L(\mathcal{G}))$.

Consider now x_i to be node (or for our case agent) v_i's state. The continuous-time consensus protocol is defined as $\dot{x}_i(t) = \sum_{\{i,j\}\in E}(x_j(t) - x_i(t))$ where agent pairs $\{i, j\} \in E$ are able to communicate. In a compact form with $x(t) \in \mathbb{R}^n$, the collective dynamics is represented as

$$\dot{x}(t) = -L(\mathcal{G})x(t), \tag{1}$$

with $L(\mathcal{G})$ being the Laplacian of the underlying interaction topology. From the definition of the graph Laplacian all rows of $L(\mathcal{G})$ sum to zero and $\lambda_1(\mathcal{G}) = 0$ with the corresponding eigenvector as $v_1 = \mathbf{1}^T = [1, \ldots, 1]^T$. Subsequently, when $\mathcal{G}$ is connected, it can be deduced that $x = \alpha\mathbf{1}$ is a unique global

[1] The notation $[V]^2$ refers to the set of two-element subsets of V.

978-1-4244-5040-4/10/$26.00

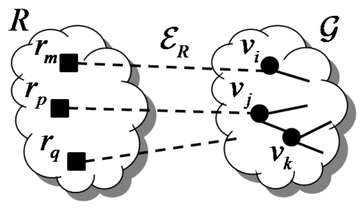
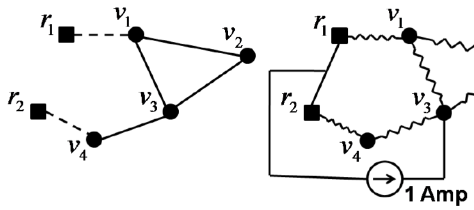


Figure 1. Example of influence notation.



$$A(\mathcal{G},\mathcal{R}) = \begin{bmatrix} -3 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -3 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}, \; B(\mathcal{R}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

Figure 2. Left: Network graph with external (control) agents r_1 and r_2 attached to agents v_1 and v_4 respectively, leading to an altered Laplacian $A(\mathcal{G},\mathcal{R})$ and input matrix $B(\mathcal{R})$ of model (2). Right: Equivalent electrical network. The potential difference $V_{v_3} - V_{\mathcal{R}}$ is the effective resistance between v_3 and common resistor node $\{r_1, r_2\}$.

exponentially-stable equilibrium of the process (1) with α as the average of the initial states [1].

We now introduce a model of *influenced* (forced) consensus associated with a control pair $\mathcal{R} = (R, \mathcal{E}_R)$, where $R \in \mathbb{R}^r$ is the control agent set and $\mathcal{E}_R \subseteq R \times V$ is the control edge set. Each control agent $r_j \in R$ is attached to exactly one $v_i \in V$ along the edge $(r_j, v_i) \in \mathcal{E}_R \in \mathbb{R}^r$ and subsequently delivers a signal $u_j(t) \in \mathbb{R}$. Figure 1 provides a graphical representation of this notation.

The resulting input-output dynamics of the influenced consensus with $u(t) \in \mathbb{R}^r$ then assumes the form,

$$\dot{x}(t) = A(\mathcal{G},\mathcal{R})x(t) + B(\mathcal{R})u(t), \tag{2}$$

where $B(\mathcal{R}) \in \mathbb{R}^{n \times r}$ with $[B(\mathcal{R})]_{ji} = 1$ when $\{r_i, v_j\} \in E_R$ and $[B(\mathcal{R})]_{ji} = 0$ otherwise, $A(\mathcal{G},\mathcal{R}) = -(L(\mathcal{G}) + M(\mathcal{R})) \in \mathbb{R}^{n \times n}$ and $M(\mathcal{R}) = B(\mathcal{R})B(\mathcal{R})^T \in \mathbb{R}^{n \times n}$. We also define the special type of single-agent control as $\mathcal{R}^i$ where $R = \{r_1\}$ and $\mathcal{E}_R = \{r_1, v_i\}$, similarly we denote double-agent control as $\mathcal{R}^{i,j}$.

We recognize $A(\mathcal{G},\mathcal{R})$ as the matrix-weighted Dirichlet, or grounded Laplacian [11], [12]. The spectrum of $A(\mathcal{G},\mathcal{R})$ relates closely to the spectrum of $L(\mathcal{G})$. In this way, the structure of underlying graph can be related to the dynamics of the model. The following highlights this connection:

Proposition 1 (System Bounds): The eigenvalues of the system matrix $-A(\mathcal{G},\mathcal{R})$ of model (2) using r control agents, place the following bounds on the eigenvalues of the graph Laplacian $L(\mathcal{G})$;

a) For all r: $\lambda_i(L(\mathcal{G})) \leq \lambda_i(-A(\mathcal{G},\mathcal{R}))$.
b) For all r:$\lambda_i(-A(\mathcal{G},\mathcal{R})) \leq \lambda_i(\mathcal{G}) + 1$.
c) For $r < i$:$\lambda_{i-r}(-A(\mathcal{G},\mathcal{R})) \leq \lambda_i(\mathcal{G})$.

Proof: By the Interlacing theorem [13] bounds a) and c) follow. By Weyl's theorem [13], $\lambda_i(-A(\mathcal{G},\mathcal{R})) = \lambda_i(L(\mathcal{G}) + M(\mathcal{R})) \leq \lambda_i(\mathcal{G}) + \lambda_n(M(\mathcal{R})) = \lambda_i(\mathcal{G}) + 1$. ∎

III. Control Influence

We now propose a control scheme for consensus-type coordination algorithms. The external (control) agents steer the network with a signal of constant expected value, e.g., white noise with a given mean. Its performance can be related to the underlying graph topology. We then provide protocols that involve localized edge swaps to encourage or deter the performance of this control scheme. We assume that $\mathcal{R}$ is composed solely of friendly or unfriendly nodes at any one instance.

We introduce an open loop control that adopts a naive approach to network control. The control agents merely attempt to steer the system to a common state u_c, deterministically or in the mean. We note that when the network is driven by a stochastic signal with a constant expected value $\mathbf{E}(u) = u_c\mathbf{1}$, the expected value of the agent states $\mathbf{E}(x)$ can be modeled as a standard consensus problem with $u = u_c\mathbf{1}$ as

$$\frac{d}{dt}\mathbf{E}(x) = \mathbf{E}(Ax + Bu) = A\mathbf{E}(x) + Bu_c\mathbf{1}.$$

Hence, influencing the network with a random signal with a constant expected value is equivalent- in the mean- with influencing the network with a constant signal.

Before continuing, we state an auxiliary result:

Proposition 2 (Negative definite): [14] The matrix $A(\mathcal{G},\mathcal{R})$ of model (2) is negative definite (and so invertible) if the original graph is connected.

We note that the state $u_c\mathbf{1}$ is reachable for the influenced system (2) as $A(\mathcal{G},\mathcal{R})$ is negative definite (Proposition 2). This highlights one advantage of the control scheme in that it requires no knowledge of network agent's states or topology, only the assumption that the network is connected, for convergence to occur. In our subsequent discussion, we will only consider connected graphs as disconnected graphs can be analyzed as the union of their connected components.

A. Average Convergence

Next, we examine the cost from uniform control u_c, over an infinite horizon, for steering the network to consensus from an arbitrary initialization. In this case, the convergence cost, with coordinate transform $\tilde{x}(t) = x(t) - u_c\mathbf{1}$, is[2]

$$2\int_0^\infty \tilde{x}(t)^T\tilde{x}(t)dt = -\tilde{x}(0)^T A(\mathcal{G},\mathcal{R})^{-1}\tilde{x}(0).$$

In order to parametrize the resilience of a consensus-type network from specific control set $\mathcal{R}$ delivering constant mean control, let us define the average cost $J^{\text{avg}}(\mathcal{G},\mathcal{R})$ as

$$\begin{aligned} J^{\text{avg}}(\mathcal{G},\mathcal{R}) &= \mathbb{E}_{\|\tilde{x}(0)\|=1}\left(-\tilde{x}(0)^T A(\mathcal{G},\mathcal{R})^{-1}\tilde{x}(0)\right) \\ &= \mathbf{tr}\left(-A(\mathcal{G},\mathcal{R})^{-1}\right)/n \\ &= \sum_{i=1}^n 1/\lambda_i\left(-A(\mathcal{G},\mathcal{R})\right)/n. \end{aligned}$$

It has previously been established that the diagonal of $-A(\mathcal{G},\mathcal{R})^{-1}$ has a resistive electrical network interpretation [11]. The agents V and R represent connection points between resistors corresponding to the communication edges E and $\mathcal{E}_R$. In addition all connection points corresponding

[2]The scaling by 2 is for presentation simplicity.

to R are shorted together. The effective resistance between two connection points in an electrical network is defined as the potential drop between the two points, when a current source with intensity equal to 1 Ampere is connected across the two points. The i-th diagonal element of $-A(\mathcal{G},\mathcal{R})^{-1}$ is the effective resistance $E_{\text{eff}}(v_i)$ between the common shorted control agents R and v_i. An example of the equivalent electrical network is displayed in Figure 2. The implication is that

$$nJ^{\text{avg}}(\mathcal{G},\mathcal{R}) = \textbf{tr}\left(-A(\mathcal{G},\mathcal{R})^{-1}\right) = \sum_{i=1}^{n} E_{\text{eff}}(v_i). \quad (3)$$

B. Average Convergence Protocols for Trees

Further generalizations can be made for the family of tree graphs $\mathcal{T}$. Tree graphs are often adopted for agent-to-agent communication topologies as they minimize edge (communication) costs while maintaining connectivity. First let us define some properties of $J^{\text{avg}}(\mathcal{G},\mathcal{R})$ individual to trees.

Let us define the special set of agents that lie on any of the shortest paths between agents in $\mathcal{R}$ as *main path agents* defined by set $\mathcal{M}$. This is a unique set for a given pair $(\mathcal{G},\mathcal{R})$. For all $v_i \notin \mathcal{M}$ there exists an unique $v_j \in \mathcal{M}$ that has a shorter minimum path to v_i than any other agent in $\mathcal{M}$, we define this agent as $\Gamma(v_i)$, i.e., $\Gamma(v_i)$ is the closest agent to v_i that is also a member of the main path. Therefore for tree graphs we can state that:

Lemma 1 (Average convergence): For the n-agent connected tree $\mathcal{T}$, the average convergence is $J^{\text{avg}}(\mathcal{T},\mathcal{R}) = (\sum_{v_i\in\mathcal{M}} E_{\text{eff}}(v_i) + \sum_{v_i\notin\mathcal{M}}\left[E_{\text{eff}}(\Gamma(v_i)) + d(v_i,\Gamma(v_i))\right])/n$.

Proof: If $v_i \notin \mathcal{M}$ then the equivalent electrical network involving v_i can be simplified into a resistor representing $E_{\text{eff}}(\Gamma(v_i))$ ohms in series with $d(v_i,\Gamma(v_i))\times 1$ ohm resistors. The result then follows from (3). ■

There is an intuitive link between the centrality of an agent in a network and its influence on the network's dynamics. This correlation becomes apparent for tree graphs in the following:

Corollary 1 (Single-control average convergence): For the n-agent connected tree $\mathcal{T}$ the average convergence of a control agent attached to any agent v_i is $J^{\text{avg}}(\mathcal{T},\mathcal{R}^i) = \left(\sum_{j=1}^{n} d(v_i,v_j) + n\right)/n$.

Proof: Follows from Lemma (1) with $v_i = \mathcal{M}$ and $E_{\text{eff}}(v_i) = 1$. ■

Corollary 2 (Single-control average convergence bounds): For the n-agent connected tree $\mathcal{T}$ the average convergence of a control agent attached to any agent v_i is bounded as $2 - 1/n \leq J^{\text{avg}}(\mathcal{T},\mathcal{R}^i) \leq (n+1)/2$.

Proof: Over all trees, the central node of the star graph has the smallest accumulative distance of $n-1$ to all other nodes and an end node of the path graph has the largest accumulative distance of $\sum_{i=1}^{n-1} i$ to all other nodes. ■

We now can propose a pair of protocols over a tree graph $\mathcal{T}$ to locally trade edges (communication links) between adjacent agents with the objective to deter or encourage the influence of control agents attached to the network and feeding in a constant mean signal. We consider a scenario where agents connected to R broadcast acknowledgment signals informing

Protocol 1 Convergence decrease edge swap

foreach *Agent* v_i **do**
  if $\exists v_j, v_k \in \mathcal{N}(v_i)$, $v_j \neq v_k$ *and* $v_j, v_k \notin \mathcal{I}(v_i)$ **then**
    $E \to E - \{v_i, v_j\} + \{v_j, v_k\}$
  end
end

Protocol 2 Convergence increase edge swap

foreach *Agent* v_i **do**
  if $v_k = \mathcal{I}(v_i)$, $\exists v_j \in \mathcal{N}(v_i)$ *and* $v_j \neq v_k$ **then**
    $E \to E - \{v_i, v_j\} + \{v_j, v_k\}$
  end
end

the network that they are being favorably or unfavorably influenced and so all agents within the graph are aware of the local directions of the control agents and more specifically their neighboring agents that are closest to agents in R. We denote these agents in the set $\mathcal{I}(v_i)$ for agent v_i. We clarify that R is solely composed of friendly or unfriendly agents and agents are able to distinguish between the control agents intent. The following lemma can be executed concurrently and/or in a random agent order, and guarantees that $J^{\text{avg}}(\mathcal{T},\mathcal{R})$ increases, and a connected tree is maintained at each iteration. We denote edge removal and addition by the set notation "$-/+$."

Lemma 2 (Edge Swap for Convergence): Consider a tree graph $\mathcal{T}_1 = (V, E_1)$, if a new graph $\mathcal{T}_2 = (V, E_2)$ is formed where $E_2 = E_1 - \{v_i, v_j\} + \{v_j, v_k\}$, $v_i, v_j, v_k \in V$, $v_j, v_k \in \mathcal{N}(v_i)$, $v_j \neq v_k$ and $v_j, v_k \notin \mathcal{I}(v_i)$; then $J^{\text{avg}}(\mathcal{T}_1,\mathcal{R}) < J^{\text{avg}}(\mathcal{T}_2,\mathcal{R})$.

Proof: If $v_m \in \mathcal{M}$ then for all $v_l \in \mathcal{N}(v_m)$ we have $v_m \in \mathcal{I}(v_l)$. Therefore in regard to the lemma $v_j, v_k \notin \mathcal{M}$. Then from Lemma 1 we have $E_{\text{eff}}(v_j) = E_{\text{eff}}(v_k) = E_{\text{eff}}(v_i) + 1$ after the edge flip $E_{\text{eff}}(v_j) = E_{\text{eff}}(v_i) + 2$ and all agents downstream of v_j effective resistance increases by 1 as well. Therefore for all agents in $\mathcal{T}_2$, the respective effective resistance increases, e.g. v_j, or stays the same and so $J^{\text{avg}}(\mathcal{T}_2,\mathcal{R}) > J^{\text{avg}}(\mathcal{T}_1,\mathcal{R})$. ■

The decentralized unfriendly Protocol 1 that endeavors to increase $J^{\text{avg}}(\mathcal{T},\mathcal{R})$ follows from Lemma 2. For all single-agent controlled trees the graph will eventually reach the greatest $J^{\text{avg}}\left(\mathcal{T},\mathcal{R}^1\right) = (n+1)/2$ corresponding to a path graph with the control agent at an end. All other graphs will acquire a path-like appearance with the main path unaffected by the protocol's edge swaps.

The protocol was applied to a random tree graph on 40 agents with a single control agent connected to v_1. The path graph with the control agent attached to the end was achieved after 100 edge flips. A sample of the intermediate graphs and the bound of convergence over all iterations are displayed in Figures 3 and 4, respectively. The cost $J^{\text{avg}}\left(\mathcal{T},\mathcal{R}^1\right)$ increased for each edge flip and no more edges flips were possible when the tree became a path graph with $J^{\text{avg}}\left(\mathcal{T},\mathcal{R}^1\right) = 20.5$.

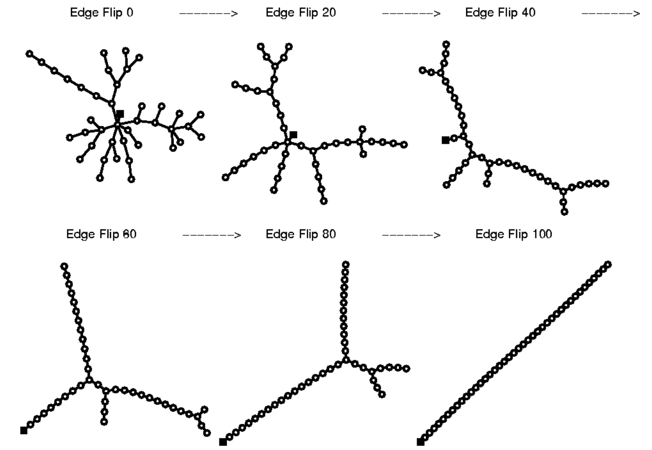


Figure 3. Selected iterations of a random tree graph with an external agent attached (square) running Protocol 1.

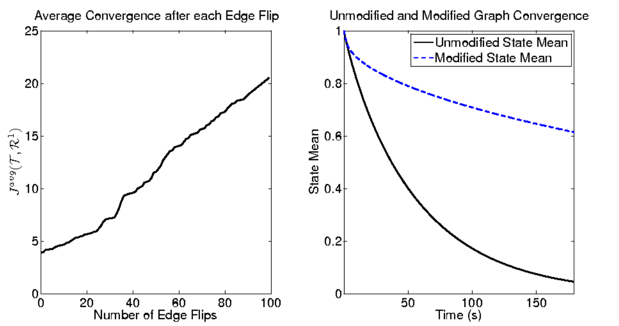


Figure 4. Left: Average convergence after each edge flip. Right: Mean of the states for the modified and unmodified graph over time for the 40 agent random tree graph running Protocol 1.

A complementary friendly Protocol 2 that aims to decrease $J^{\text{avg}}(\mathcal{T}, \mathcal{R})$ can also be obtained from Lemma 2. Under this protocol the graph converges to a star-like graph while conserving the underlying structure of the main path. The protocol was run on a 40 agent random tree graph with 3 control agents. The original and final graphs are displayed in Figure 5.

Remark 1: Under local knowledge, Lemma 2 describes the only edge swaps that guarantee J^{avg} increases. Let us examine a scenario where more than one main path agent is involved in an edge swap - a situation not included in Lemma 2, i.e., swaps involving $v_j \in \mathcal{I}(v_i)$ and/or $v_k \in \mathcal{I}(v_i)$. Consider the tree graph $\mathcal{T}_1 = (V, E_1)$ displayed in Figure 6a). We note that
$J^{\text{avg}}(\mathcal{T}_1, \mathcal{R}^{1,5}) = \frac{1}{n}\left(\frac{35}{6} + \frac{5}{6}p + \frac{8}{6}q + \frac{p}{2}(p+1) + \frac{q}{2}(q+1)\right).$

Let us consider the potential edge swaps available to agent $v_2 \in \mathcal{M}$. Locally, agent v_2 is aware that $\mathcal{N}(v_2) = \{z_1, v_1, v_3\}$ and $\mathcal{I}(v_2) = \{v_1, v_3\}$, the potential edge swaps cases are:

1) One neighbor on and one off the main path e.g. $E_2 = E_1 - \{v_2, z_1\} + \{z_1, v_1\}$ forming $\mathcal{T}_2 = (V, E_2)$ and $E_3 = E_1 - \{v_2, z_1\} + \{z_1, v_3\}$ forming $\mathcal{T}_3 = (V, E_3)$.

2) Both neighbors on the main path e.g. $E_4 = E_1 - \{v_2, v_3\} + \{v_3, v_1\}$ forming $\mathcal{T}_4 = (V, E_4)$.

Case 1: Here,
$J^{\text{avg}}(\mathcal{T}_2, \mathcal{R}^{1,5}) = \frac{1}{n}\left(\frac{35}{6} + \frac{5}{6}p + \frac{9}{6}q + \frac{p}{2}(p+1) + \frac{q}{2}(q+1)\right)$

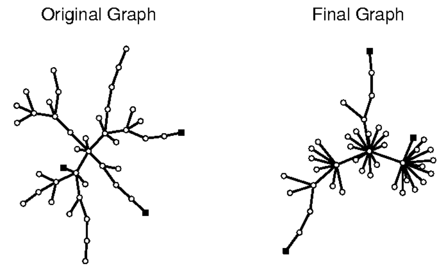


Figure 5. Original random tree and final graph with three external agents attached (squares) after applying Protocol 2.

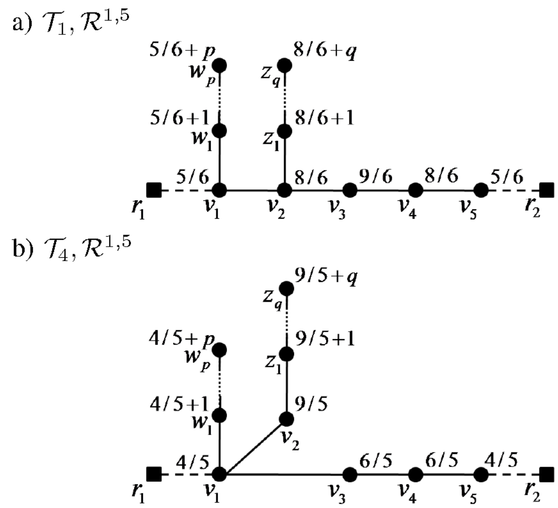


Figure 6. Two tree graphs with two attached control agents $\{r_1, r_2\}$. The effective resistance $E_{eff}(v_i)$ appears adjacent to each agent.

$J^{\text{avg}}(\mathcal{T}_3, \mathcal{R}^{1,5}) = \frac{1}{n}\left(\frac{35}{6} + \frac{5}{6}p + \frac{5}{6}q + \frac{p}{2}(p+1) + \frac{q}{2}(q+1)\right)$
$J^{\text{avg}}(\mathcal{T}_3, \mathcal{R}^{1,5}) < J^{\text{avg}}(\mathcal{T}_1, \mathcal{R}^{1,5}) < J^{\text{avg}}(\mathcal{T}_2, \mathcal{R}^{1,5})$

As under local information v_1 and v_3 are indiscernible, case 1 does not guarantee that J^{avg} is increasing or decreasing.

Case 2: Graph $\mathcal{T}_4$ is displayed in Figure 6b) with corresponding average convergence cost,
$J^{\text{avg}}(\mathcal{T}_4, \mathcal{R}^{1,5}) = \frac{1}{n}\left(\frac{29}{5} + \frac{4}{5}p + \frac{9}{5}q + \frac{p}{2}(p+1) + \frac{q}{2}(q+1)\right)$
$J^{\text{avg}}(\mathcal{T}_1, \mathcal{R}^{1,5}) - J^{\text{avg}}(\mathcal{T}_4, \mathcal{R}^{1,5}) = \frac{1}{30n}(1 + p - 14q)$
So if $p > 14q - 1$ then $J^{\text{avg}}(\mathcal{T}_1, \mathcal{R}^{1,5}) > J^{\text{avg}}(\mathcal{T}_4, \mathcal{R}^{1,5})$ and for $p < 14q - 1$ then $J^{\text{avg}}(\mathcal{T}_1, \mathcal{R}^{1,5}) < J^{\text{avg}}(\mathcal{T}_4, \mathcal{R}^{1,5})$. Under only local knowledge the relative magnitudes of p and q cannot be discerned so no guarantees may be assumed.

A by product of this remark is that a strictly increasing local-knowledge protocol cannot guarantee the tree graph with the largest J^{avg} for $r > 1$ control agents.

C. Noisy and Impulse Control

It is not uncommon that the mean is not of central interest to a problem and adjustment of the variance of the states may be desired. Further, motivated by devious intrusion type techniques that may employ pulse-like control to avoid triangulation, the energy of the states from an unit impulse input is another potentially desirable indicator. With this in mind, the controllability gramian, defined as $P =: \int_0^\infty e^{A\tau} B B^T e^{A^T\tau} d\tau$, proves to be particularly suitable for such an analysis. We will focus on, $\mathbf{tr}(P)$ as

a) the average variance is $1/n\sum_{i=1}^{n}\mathbb{E}\left(\tilde{x}_i^2(t)\right) = (1/n)\,tr\left(\mathbb{E}\left[\tilde{x}(t)\tilde{x}^T(t)\right]\right) = (1/n)\,tr(P)$ as $t\to\infty$ over n outputs due to a unit intensity white noise input.

b) the energy of the states at the output from an unit impulse input is $\int_0^\infty x^T(t)x(t)dt = tr(P)$.

We note that P will be dependent on $\mathcal{G}$ and $\mathcal{R}$ and so henceforth is denoted by $P(\mathcal{G},\mathcal{R})$. We have,

$$\begin{aligned}\mathbf{tr}(P(\mathcal{G},\mathcal{R})) &= \mathbf{tr}\left(\int_0^\infty e^{A(\mathcal{G},\mathcal{R})\tau}BB^Te^{A(\mathcal{G},\mathcal{R})^T\tau}d\tau\right)\\ &= \mathbf{tr}\left(BB^T\int_0^\infty e^{2A(\mathcal{G},\mathcal{R})\tau}d\tau\right)\\ &= -(1/2)\,\mathbf{tr}\left(BB^TA(\mathcal{G},\mathcal{R})^{-1}\right).\end{aligned}$$

Lemma 3 (General: $\mathbf{tr}(P)$*):* For connected graphs and the influence model (2) $\mathbf{tr}(P(\mathcal{G},\mathcal{R})) = \frac{1}{2}\sum_{\{v_i,r_j\}\in\mathcal{E}_R}E_{\text{eff}}(v_i)$.

Proof: We note that BB^T is a purely diagonal matrix with $\left[BB^T\right]_{ii} = 1$ if $\{v_i,r_j\}\in\mathcal{E}_R$ and $\left[BB^T\right]_{ii} = 0$, otherwise. Therefore $\left[BB^TA(\mathcal{G},\mathcal{R})^{-1}\right]_{ii} = \left[A(\mathcal{G},\mathcal{R})^{-1}\right]_{ii}$ if $\{v_i,r_j\}\in\mathcal{E}_R$ and $\left[BB^TA(\mathcal{G},\mathcal{R})^{-1}\right]_{ii} = 0$, otherwise. The statement of the lemma now follows. ■

Corollary 3 (Single-control: $\mathbf{tr}(P)$*):* For connected graphs and the influence model (2) with one control agent, $\mathbf{tr}(P(\mathcal{G},\mathcal{R})) = 1/2$.

Proof: The effective resistance of $\{v_i,r_1\} = \mathcal{E}_R$ is $E_{\text{eff}}(v_i) = 1$ as there is only one resistor link between v_i and r_1. The corollary follows. ■

The implication of Corollary 3 is that on the average, *a single-agent controlled n-agent connected graph has the same reduction in average variance to white noise and energy dissipation from an impulse input regardless of the structure of the network and where the control agent is connected.*

D. Noisy and Impulse Control Protocols for Trees

We propose another protocol for tree graphs $\mathcal{T}$ now with the objective of reducing the state variance due to control agents attached to the network and feeding in unit intensity white noise. Again the protocol involve local edge trade executed concurrently and/or in a random agent order which guarantees that $\mathbf{tr}(P(\mathcal{T},\mathcal{R}))$ decreases and a connected tree is maintained at each iteration. A complementary protocol to increase the average state variance is also proposed.

We note for a connected tree graph $\mathbf{tr}(P(\mathcal{T},\mathcal{R}))$ is only dependent on $d(r_i,r_j)$ for all $\{r_i,r_j\}$ pairs in R, and so only dependent on the graph of the main path with agents $\mathcal{M}$.

Lemma 4 (Edge Swap for Variance): Consider a tree graph $\mathcal{T}_1 = (V,E_1)$. If a new graph $\mathcal{T}_2 = (V,E_2)$ is formed where $E_2 = E_1 - \{v_i,v_j\} + \{v_j,v_k\}$, $v_i,v_j,v_k\in V$, $v_j,v_k\in\mathcal{N}(v_i)$, $v_j\neq v_k$ and $v_j,v_k\in\mathcal{I}(v_i)$ then $\mathbf{tr}(P(\mathcal{T}_1,\mathcal{R}))\geq\mathbf{tr}(P(\mathcal{T}_2,\mathcal{R}))$.

Proof: Firstly, as v_i has $|\mathcal{I}(v_i)| = 2$ then $v_i\in\mathcal{M}$. As v_j and v_k are closer to a control agent than the main path agent v_i then $v_j,v_k\in\mathcal{M}$. The edge swap involves removing v_i from $\mathcal{M}$. As $\mathbf{tr}(P(\mathcal{T},\mathcal{R}))$ only depends on the subgraph involving only agents of $\mathcal{M}$ then the effect is to reduce an edge resistance within the electrical network representing this subgraph. Rayleigh's Monotonicity Law states that if the edge resistance in a regular electrical network is decreased then the effective resistance between any two agents in the network can only decrease. This result was extended to generalized electrical networks. Therefore, $\sum_{\{v_i,r_j\}\in\mathcal{E}_R}E_{\text{eff}}(v_i)$ will not increase and the lemma follows. ■

Protocol 3 Average variance reduction edge swap

foreach *Agent* v_i **do**
  if $\exists v_j,v_k\in\mathcal{N}(v_i)$, $v_j\neq v_k$ *and* $v_j,v_k\in\mathcal{I}(v_i)$ **then**
    $E\to E-\{v_i,v_j\}+\{v_j,v_k\}$
  end
end

Protocol 4 Unit impulse detection edge swap

foreach *Agent* v_i **do**
  if $|\mathcal{I}(v_i)| > 1$ *and* $\exists v_j,v_k\in\mathcal{N}(v_i)$, $v_j\in\mathcal{I}(v_i)$ *and* $v_k\notin\mathcal{I}(v_i)$ **then**
    $E\to E-\{v_i,v_j\}+\{v_j,v_k\}$
  end
end

Protocol 3 that decreases $\mathbf{tr}(P(\mathcal{T},\mathcal{R}))$ follows from this lemma. Single-agent controlled trees will remain unaffected by this protocol. For double-agent controlled trees the main path will degenerate to $\{v_i,v_j\} = \mathcal{M}$ where $(\{v_i,r_1\},\{v_j,r_2\}) = \mathcal{E}_R$. Protocol 3 was run on a 40 agent random tree with 3 control agents the original and final graphs are displayed in Figure 7.

A complementary energy amplification Protocol 4, that aims to increase $\mathbf{tr}(P(\mathcal{T},\mathcal{R}))$, can also be obtained from Lemma 2. This protocol is suitable for impulse detection as larger $\mathbf{tr}(P(\mathcal{T},\mathcal{R}))$ produces higher output energy $\int_0^\infty x^T(t)x(t)dt$.

Remark 2: For $|\mathcal{I}(v_i)| > 2$, an edge swap has the effect of reducing v_i's degree and elongate the main path subgraph. Rayleigh's Monotonicity Law cannot be applied in this scenario as no resistance is being removed from the main path. Similar to the remark in the previous section these edge swaps do not guarantee $\mathbf{tr}(P(\mathcal{T}_1,\mathcal{R}))\geq\mathbf{tr}(P(\mathcal{T}_2,\mathcal{R}))$. Therefore the posed protocols are the best local-knowledge edge swapping methods and no guarantees can be made that the local-knowledge method will converge to the global-knowledge edge swap solution.

We previously remarked that Protocols 1 and 2 do not alter the main path. Consequently, by Lemma 3, the $\mathbf{tr}(P(\mathcal{T},\mathcal{R}))$ is conserved throughout these protocols so that, eventhough the mean convergence is altered, the steady state variance remains the same. The converse is not true as Protocols 3 and 4 involve manipulations of the main path and, as remarked in the previous section, this can arbitrarily vary $J^{\text{avg}}(\mathcal{T},\mathcal{R})$. Generally speaking as $\mathbf{tr}(P(\mathcal{T},\mathcal{R}))$ increases under Protocol 4 the graphs are elongated and so $J^{\text{avg}}(\mathcal{T},\mathcal{R})$ tends to increase. Similarly as $\mathbf{tr}(P(\mathcal{T},\mathcal{R}))$ decreases under Protocol 3 the graphs compress and so $J^{\text{avg}}(\mathcal{T},\mathcal{R})$ tends to decrease. The extreme examples, for a given r, are star graphs with smallest $J^{\text{avg}}(\mathcal{T},\mathcal{R})$ and path graphs with largest $J^{\text{avg}}(\mathcal{T},\mathcal{R})$.

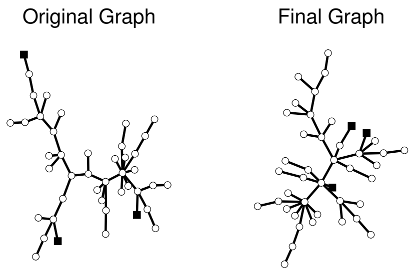


Figure 7. Original random tree and final graph with three external agents attached (squares) after applying Protocol 3.

IV. Example: Clock Synchronization

Clock synchronization is often necessary in many distributed systems improving the consistency of data and the correctness of algorithms. Precise time synchronization is needed for distributed application such as sensor data fusion, scheduling, localization, coordinated actuation and power-saving duty cycling. Motivated by the work of [5] we assembled the following experiment.

Consensus on clock time was run on 100 decentralized computer terminals (our agents) communicating over a tree network. Because time consensus can only correct for differential errors between terminals, not absolute errors without a reference, friendly control agents periodically connect to the network and deliver the constant correction for the absolute bias in the system. On connection, the friendly control agents initiate a friendly flag which is passed through the network, providing the local direction of the friendly agents and initiating Protocol 2. The network adapts under this protocol to promote convergence to the correct absolute clock time. On disconnection, the agents initiate a disconnect flag.

Similarly, we introduce a malicious control agent that attempts to drive the system to a false absolute time. On connection the control adjacent agents send out a distress signal triggering the network to initiate Protocol 1 so as to deter the false convergence of the network. It is assumed that the friendly agents on discovery of an malicious control agent will clear the network of these foreign agents and trigger the termination of Protocol 1 before commencing delivery of the correction signal again. In other words friendly and malicious agents would not be concurrently connected to the network.

To examine the effectiveness of the protocols, equal time was provided for both friendly and malicious control agents, specifically alternating 10 sec intervals for 800 sec. This switching interval is long enough for transients to settle and so is appropriate for the protocols. The network was initialized in a random tree with all agents at the time offset of 0 sec (the correct offset is 1 sec). The set $\mathcal{E}_R$ formed over 3 control agents is randomly selected at each new 10 sec interval. The friendly and unfriendly control agents deliver time offsets of 1 sec and 0 sec, respectively. The average of the constant values, i.e. 0.5 sec, would be expected for the mean offset without the protocols. The protocols are able to favor the friendly agent bringing the average offset to 0.59 sec. Clock offset means are displayed for the unmodified and modified graphs in Figure 8.

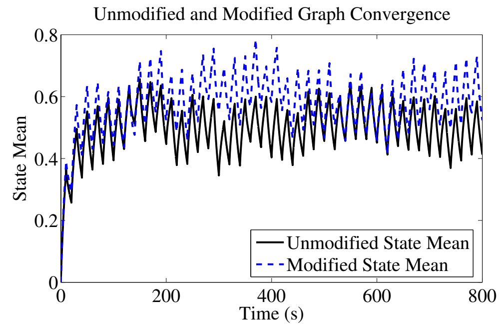


Figure 8. The state means (clock offset means) of the unmodified and modified tree graphs over time.

V. Conclusion

This paper presents a formulation of a constant mean control in semi-autonomous networks. The mean and variance were examined as they relate to an electrical analogy of the network. Four decentralized protocols were proposed for tree graphs to increase or decrease convergence and variance within the network. The framework provides a setting for reasoning about amenability of coordination algorithms to external signals and agents. It also identifies critical graph-theoretic parameters that can influence the *synthesis of network geometries* that support the operation of multi-agent semi-autonomous networks.

References

[1] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and Cooperation in Networked Multi-Agent Systems," *Proceedings of the IEEE*, vol. 95, pp. 215–233, 2007.

[2] H. G. Tanner, G. J. Pappas, and V. Kumar, "Leader-to-formation stability," *IEEE Transactions on Robotics and Automation*, vol. 20, pp. 443–455, June 2004.

[3] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Transactions on Automatic Control*, vol. 48, pp. 988–1001, June 2003.

[4] Y. Hatano and M. Mesbahi, "Agreement over random networks," *IEEE Transactions on Automatic Control*, vol. 50, pp. 1867–1872, 2005.

[5] S. Graham and P. Kumar, "Time in general-purpose control systems: the Control Time Protocol and an experimental evaluation," *43rd IEEE Conference on Decision and Control*, pp. 4004–4009 Vol.4, 2004.

[6] A. Gueye and J. C. Walrand, "Security in networks: A game-theoretic approach," in *Proc. 47th IEEE Conference on Decision and Control*, 2008, pp. 829–834.

[7] M. Bloem, T. Alpcan, and T. Basar, "Optimal and robust epidemic response for multiple networks," in *Proc. 46th IEEE Conference on Decision and Control*, 2007, pp. 5074–5079.

[8] R. Olfati-Saber, "Flocking for multi-agent dynamic systems: algorithms and theory," *IEEE Transactions on Automatic Control*, vol. 51, pp. 401–420, March 2006.

[9] A. Fagiolini, G. Valenti, L. Pallottino, G. Dini, and A. Bicchi, "Decentralized intrusion detection for secure cooperative multi-agent systems," in *Proc. 46th IEEE Conference on Decision and Control*, 2007, pp. 1553–1558.

[10] F. Pasqualetti, A. Bicchi, and F. Bullo, "Distributed intrusion detection for secure consensus computations," in *Proc. 46th IEEE Conference on Decision and Control*, 2007, pp. 5594–5599.

[11] P. Barooah and J. P. Hespanha, "Graph Effective Resistance and Distributed Control: Spectral Properties and Applications," in *Proc. 45th IEEE Conference on Decision and Control*, 2006, pp. 3479–3485.

[12] S. Salsa, *Partial differential equations in action: from modelling to theory*. New York: Springer Verlag, 2008.

[13] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge University Press, N Y, 1990.

[14] A. Chapman, M. Nabi-Abdolyousefi, and M. Mesbahi, "Identification and Infiltration in Consensus-type Networks," *1st IFAC Workshop on Estimation and Control of Networked Systems*, 2009.