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Semi-Autonomous Networks: Network Resilience and Adaptive Trees

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***Abstract*—This paper examines the dynamics of a networked multi-agent system operating with a consensus-type coordination algorithm that can be influenced by external agents. We refer to this class of networks as *semi-autonomous*. Within such a class, we consider a network's resilience to the influence of external agents delivering a test signal, namely a Gaussian noise with a given mean. Specifically, we examine the resultant mean and variance of the states of the agents in the network, via metrics dubbed as mean and variance resilience, as well as relate these quantities to circuit-theoretic notions of the network. These metrics are then used to propose adaptive protocols for tree graphs to increase or decrease the mean and variance resilience. Finally, a hybrid protocol is proposed which is shown to have a guaranteed performance using game-theoretic techniques. All protocols involve decentralized edge swaps that can be performed in parallel, asynchronously, and require only local agent information of the graph structure.**

***Index Terms*—Semi-autonomous networks; Consensus protocol; Effective Resistance; Coordinated control over networks**

I. Introduction

Consensus-type algorithms provide effective means for distributed information-sharing and control for networked, multi-agent systems in settings such as multi-vehicle control, formation control, swarming, and distributed estimation; see for example, [1], [2], [3], [4]. An appeal of consensus algorithms is their ability to operate *autonomously* over simple trusting agents. This has the added benefit that external (control) agents, perceived as *native* agents, can seamlessly attach to the network and steer it in particular directions. These additional agents, ignoring consensus rules, will influence the system dynamics compared to the *unforced* networked system resulting in scenarios such as leader-follower [2] and drift correction [5]. The detriment is that this same approach can be adopted by malicious infiltrating agents. We refer to this class of systems, with friendly and/or unfriendly attached nodes, as *semi-autonomous* networks.

In this paper we examine the resilience of a network to the effect of an external agent injecting a *test signal*, namely a white Gaussian signal, into the network. The resilience of the network is measured in terms of the mean and variance of the agents' state; we refer to these metrics as the *mean* and *variance resilience*. For the mean resilience, an electrical network analogy is used to measure the average cost of convergence. For the variance resilience, on the other hand, we use the controllability gramian of the resulting system. Both types of resilience are subsequently used to propose decentralized protocols for tree graphs that adaptively aim to improve or degrade the network's resilience by performing local edge swaps.

The contribution of the paper is two fold. First, we provide a system-theoretic approach to examine and reason about the resilience of influenced coordination protocols operating over a graph. Secondly, we provide distributed adaptation schemes that the agents in the network can adopt in order to alter the influence of the external agents on the network operation.

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II. Background and Model

We provide a brief background on constructs and models that will be used in this paper.

An undirected graph $\mathcal{G} = (V, E)$ is defined by a node set V with cardinality n and an edge set E comprised of pairs of nodes, where nodes v_i and v_j are adjacent if $\{v_i, v_j\} \in E$. We denote the set of nodes adjacent to v_i as $\mathcal{N}(v_i)$ and the minimum path length, induced by the graph, between nodes v_i and v_j as $d(v_i, v_j)$. The degree δ_i of node v_i is the number of its adjacent nodes. The degree matrix $\Delta(\mathcal{G}) \in \mathbb{R}^{n\times n}$ is a diagonal matrix with δ_i at position (i,i). The adjacency matrix is an $n \times n$ symmetric matrix with $[\mathcal{A}(\mathcal{G})]_{ij} = 1$ when $\{v_i, v_j\} \in E$ and $[\mathcal{A}(\mathcal{G})]_{ij} = 0$ otherwise. The combinatorial Laplacian is defined as $L(\mathcal{G}) = \Delta(\mathcal{G}) - \mathcal{A}(\mathcal{G})$.

Now consider $x_i(t) \in \mathbb{R}$ to be node (or for our case agent) v_i's state at time t. The continuous-time consensus protocol is defined as $\dot{x}_i(t) = \sum_{\{i,j\}\in E} (x_j(t) - x_i(t))$ where agent pairs $\{i, j\} \in E$ are able to communicate. In a compact form with $x(t) \in \mathbb{R}^n$, the collective dynamics is represented as $\dot{x}(t) = -L(\mathcal{G})x(t)$ with $L(\mathcal{G})$ being the Laplacian of the underlying interaction topology [1].

We next introduce a model of *influenced* consensus associated with a control pair $\mathcal{R} = (R, \mathcal{E}_R)$, where $R \in \mathbb{R}^r$ is the set of r external agent and $\mathcal{E}_R \subseteq R \times V$ is the set of edges used by the external agents to inject signals into the network. It is assumed that each external agent $r_j \in R$ is attached to exactly one node $v_i \in V$ along the edge $\{r_j, v_i\} \in \mathcal{E}_R \in \mathbb{R}^r$ and subsequently delivers a signal $u_j(t) \in \mathbb{R}$.

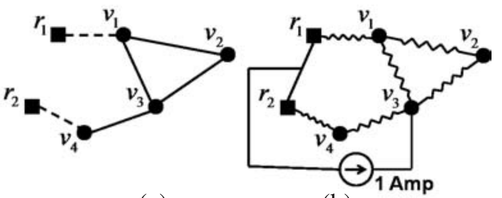
The resulting influenced system now assumes the form,

$$\dot{x}(t) = A(\mathcal{G}, \mathcal{R})x(t) + B(\mathcal{R})u(t), \tag{1}$$

where $B(\mathcal{R}) \in \mathbb{R}^{n\times r}$ with $[B(\mathcal{R})]_{ij} = 1$ when $\{r_j, v_i\} \in \mathcal{E}_R$ and $[B(\mathcal{R})]_{ij} = 0$ otherwise, and

$$A(\mathcal{G}, \mathcal{R}) = -\left(L(\mathcal{G}) + M(\mathcal{R})\right) \in \mathbb{R}^{n\times n}, \tag{2}$$

where $M(\mathcal{R}) = B(\mathcal{R})B(\mathcal{R})^T \in \mathbb{R}^{n\times n}$. We also define the special type of single-agent control as $\mathcal{R}^i$ where $R = \{r_1\}$



(a) (b)

$$A(\mathcal{G},\mathcal{R}) = \begin{bmatrix} -3 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -3 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}, B(\mathcal{R}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

Figure 1. (a) Network graph with external (control) agents r_1 and r_2 attached to agents v_1 and v_4 respectively, leading to an altered Laplacian $A(\mathcal{G},\mathcal{R})$ and input matrix $B(\mathcal{R})$ of model (1). (b) Equivalent electrical network. The potential difference $V_{v_3} - V_{\mathcal{R}}$ is the effective resistance between v_3 and common resistor node $\{r_1, r_2\}$.

and $\mathcal{E}_R = \{r_1, v_i\}$. Further the set of agents v_i such that $\{r_j, v_i\} \in \mathcal{E}_R$ for some r_j will be denoted by $\pi(\mathcal{E}_R)$.

We recognize $A(\mathcal{G},\mathcal{R})$ as the matrix-weighted Dirichlet, or grounded Laplacian [6], [7]. An auxiliary observation on the Dirichlet matrix, to be used subsequently, is the following.

Proposition 1: [8] The matrix $A(\mathcal{G},\mathcal{R})$ of model (1) is negative definite (and so invertible) if the graph is connected. In order to quantify the resilience of a network to resist the influence of external agents, the following two observations are in order: (1) the dynamics of the mean of agents' state is captured by model (1) where u is replaced by the mean of the Gaussian noise u_c, (2) when the underlying graph is connected, all agents' state converge in the mean to u_c. The last statement is a direct consequence of Proposition 1. We approach the network resilience problem from two fronts; first as a network's ability to resist its agent states' convergence to u_c - dubbed *mean resilience* (Section III) - and secondly its ability to resist its agent states' variance increasing due to a noisy external agent's signal - dubbed *variance resilience* (Section IV).

III. Mean Resilience

The mean resilience is a metric for the effectiveness of a network, via its topology, to resist convergence to the mean of external agents' signal u_c. We derive the mean resilience as the cost incurred by external agents to steer the mean of the states to u_c. More specifically, noting that $\mathbf{1} = [1, \ldots, 1]^T$, $\mathbf{1}_x \in \mathbb{R}^n$, $\mathbf{1}_u \in \mathbb{R}^r$ and $A(\mathcal{G},\mathcal{R})^{-1} B \mathbf{1}_u = -\mathbf{1}_x$, the convergence cost, with coordinate transform $\tilde{x}(t) = x(t) - u_c \mathbf{1}_x$ can be derived as,[1]

$$\begin{aligned} 2\int_0^T \tilde{x}^T \tilde{x} dt &= \int_0^T \tilde{x}^T x + x^T \tilde{x} - \tilde{x}^T \mathbf{1}_x u_c - u_c \mathbf{1}_x^T \tilde{x} dt \\ &= \int_0^T \tilde{x}^T x + x^T \tilde{x} + \tilde{x}^T A^{-1} B \mathbf{1}_u u_c + u_c \mathbf{1}_u^T B^T A^{-1} \tilde{x} dt \\ &= \int_0^T (Ax + B\mathbf{1}_u u_c)^T A^{-1} \tilde{x} + \tilde{x}^T A^{-1} (Ax + B\mathbf{1}_u u_c)\, dt \\ &= \int_0^T \dot{x}^T A^{-1} \tilde{x} + \tilde{x}^T A^{-1} \dot{x} dt = \int_0^T \frac{d}{dt} \tilde{x}(t)^T A^{-1} \tilde{x}(t) dt \\ &= \tilde{x}(T)^T A^{-1} \tilde{x}(T) - \tilde{x}(0)^T A^{-1} \tilde{x}(0). \end{aligned}$$

In order to parametrize the resilience of the network for a specific control set $\mathcal{R}$, let us define the accumulative state mean over a finite time horizon T and $\tilde{x}(0)$ uniformly distributed about the unit circle as

$$\begin{aligned} J^{\text{avg}}(\mathcal{G},\mathcal{R},T) &= \mathbb{E}_{\|\tilde{x}(0)\|=1} \left(2\int_0^T \tilde{x}(t)^T \tilde{x}(t) dt \right) \\ &= \mathbb{E}_{\|\tilde{x}(0)\|=1} \left(\tilde{x}(T)^T A^{-1} \tilde{x}(T) - \tilde{x}(0)^T A^{-1} \tilde{x}(0) \right) \\ &= \mathbb{E}_{\|\tilde{x}(0)\|=1} \mathbf{tr} \left(\tilde{x}(0)\tilde{x}(0)^T \left(\left(e^{AT}\right)^T A^{-1} e^{AT} - A^{-1} \right) \right) \\ &= \mathbb{E}_{\|\tilde{x}(0)\|^2=n} \mathbf{tr} \left(\frac{1}{\sqrt{n}} \tilde{x}(0) \frac{1}{\sqrt{n}} \tilde{x}(0)^T (e^{AT} A^{-1} e^{AT} - A^{-1}) \right) \\ &= \frac{1}{n} \mathbf{tr} \left(\left(\mathbb{E}_{\|\tilde{x}(0)\|^2=n} \tilde{x}(0)\tilde{x}(0)^T \right) \left(e^{AT} A^{-1} e^{AT} - A^{-1} \right) \right) \\ &= \frac{1}{n} \mathbf{tr}(I(e^{2AT} - I)A^{-1}) = \frac{1}{n} \sum_{i=1}^n \frac{1}{\lambda_i(-A)} (1 - e^{-2\lambda_i(-A)T}). \end{aligned}$$

We assume time T is unknown but sufficiently large, justifying the use of $J^{\text{avg}}(\mathcal{G},\mathcal{R},\infty)$ as a measure of mean resilience. In fact for brevity, we let $J^{\text{avg}}(\mathcal{G},\mathcal{R},\infty) = J^{\text{avg}}(\mathcal{G},\mathcal{R})$. We can now formally define our metric.

Definition 2: The *mean resilience* of a network is the average cost incurred by external agents to steer the mean state of the entire network to its own mean value, over an infinite horizon, and is equal to

$$J^{\text{avg}}(\mathcal{G},\mathcal{R}) = \frac{1}{n} \mathbf{tr} \left(-A(\mathcal{G},\mathcal{R})^{-1} \right).$$

The following section will provide more insight into the mean resilience.

A. Analysis of Mean resilience

It has previously been established that the diagonal of $-A(\mathcal{G},\mathcal{R})^{-1}$ has a resistive electrical network interpretation [6]. In this setup, the agents V and R, defined in Section II, represent connection points between 1 ohm resistors corresponding to the communication edges E and $\mathcal{E}_R$. In addition, all connection points corresponding to R are electrically shorted together. The effective resistance between two connection points in an electrical network is defined as the potential drop between the two points, when a 1 Amp current source is connected across the two points. The i-th diagonal element of $-A(\mathcal{G},\mathcal{R})^{-1}$ is the effective resistance $E_{\text{eff}}(v_i)$ between the common shorted external agents R and v_i. An example of the equivalent electrical network is displayed in Figure 1. The implication is that

$$J^{\text{avg}}(\mathcal{G},\mathcal{R}) = \frac{1}{n} \sum_{i=1}^n E_{\text{eff}}(v_i). \tag{3}$$

Tree graphs are often adopted for agent-to-agent communication topologies as they minimize edge (communication) costs while maintaining connectivity. We define some properties of $J^{\text{avg}}(\mathcal{G},\mathcal{R})$ specific to trees.

In this direction, let us first define the special set of agents that lie on any of the shortest paths between agents in $\mathcal{R}$ as *main path agents* designated by set $\mathcal{M}$. This is a unique set for a given pair $(\mathcal{G},\mathcal{R})$. For all $v_i \notin \mathcal{M}$ there exists an unique $v_j \in \mathcal{M}$ that has a shorter minimum path to v_i than any other agent in $\mathcal{M}$, we define this agent as $\Gamma(v_i)$, i.e., $\Gamma(v_i)$ is the closest agent to v_i that is a member of the main path. Therefore for tree graphs we can state the following.

[1]The scaling by 2 is cosmetic.

Lemma 3: [Mean resilience for trees] For the n-agent connected tree $\mathcal{T}$, the mean resilience is $J^{\text{avg}}(\mathcal{T},\mathcal{R})$

$$= \frac{1}{n}\Big(\sum_{v_i\in\mathcal{M}} E_{\text{eff}}(v_i) + \sum_{v_i\notin\mathcal{M}} \left[E_{\text{eff}}(\Gamma(v_i)) + d(v_i,\Gamma(v_i))\right]\Big).$$

Proof: If $v_i \notin \mathcal{M}$ then the equivalent electrical network involving v_i can be simplified into a resistor representing $E_{\text{eff}}(\Gamma(v_i))$ ohms in series with $d(v_i,\Gamma(v_i))\times 1$ ohm resistors. The result then follows from (3). ■

There is an intuitive link between the *centrality* of an agent in a network and its influence on the network's dynamics. This correlation becomes apparent for tree graphs in the following.

Corollary 4: [Single-external mean resilience] For the n-agent connected tree $\mathcal{T}$ the mean resilience of the network to a single external agent attached to any agent $v_i \in V$ is

$$J^{\text{avg}}(\mathcal{T},\mathcal{R}^i) = \frac{1}{n}\Big(\sum_{j=1}^{n} d(v_i,v_j) + n\Big).$$

Proof: Follows from Lemma 3 with $v_i = \mathcal{M}$ and $E_{\text{eff}}(v_i) = 1$. ■

Corollary 5: [Single-external mean resilience bounds] For the n-agent connected tree $\mathcal{T}$ the mean resilience of the network to a single external agent attached to any agent $v_i \in V$ is bounded as $2 - \frac{1}{n} \leq J^{\text{avg}}(\mathcal{T},\mathcal{R}^i) \leq \frac{1}{2}(n+1)$.

Proof: Over all trees, the central node of the star graph has the smallest accumulative distance of $n-1$ to all other nodes and an end node of the path graph has the largest accumulative distance of $\sum_{i=1}^{n-1} i$ to all other nodes. ■

Proposition 6: [Multi-external mean resilience bounds] For the n-agent connected tree $\mathcal{T}$ the mean resilience of r external agents attached to any set of agents in V is bounded above by a graph with all main path nodes satisfying $v_i \in \pi(\mathcal{E}_R)$ and $J^{\text{avg}}(\mathcal{T},\mathcal{R}) \leq \frac{1}{2n}\left((n-r)^2 + 3(n-r) + r + 2/(r+1)\right)$.

Proof: From our effective resistance interpretation of $J^{\text{avg}}(\mathcal{T},\mathcal{R})$ (3), we note that adding resistors in series generates a higher resistance than adding in parallel. Therefore, $\text{argmax}_{(\mathcal{T},\mathcal{R})} J^{\text{avg}}(\mathcal{T},\mathcal{R})$ implies $\mathcal{M} = \pi(\mathcal{E}_R)$. Furthermore from Lemma 3, the largest accumulative distance for $v_i \notin \mathcal{M}$ will correspond to a path connected to the highest effective resistance node of $\mathcal{M}$. Now the main path subgraph with the highest effective resistance sum is the star graph with the least number of parallel resistors. Applying resistor rules we find that the star graph $\mathcal{S}$ with an external agent connected to each node, i.e., $r = n$ is the largest mean resilience graph with $J^{\text{avg}}(\mathcal{S},\mathcal{R}) = \left(r^2+r+2\right)/2r(r+1)$. Similarly, the effective resistance of an agent in the main path subgraph is $E_{\text{eff}}(v_i) \leq 1$ as the equivalent electric network is a parallel resistor cascade of 1Ω resistors. Combining this bound and the main path subgraph $\mathcal{S}$, we have
$J^{\text{avg}}(\mathcal{T},\mathcal{R}) \leq \frac{1}{n}(rJ^{\text{avg}}(\mathcal{S},\mathcal{R}) + \sum_{i=1}^{n-r}\left(E_{\text{eff}}(v_1)+i\right))$
$\leq \frac{1}{2n}((n-r)^2 + 3(n-r) + r + 2/(r+1))$. ■

B. Adaptive Protocol to Improve the Mean Resilience for Trees

We now can propose a protocol over a tree graph $\mathcal{T}$ to locally trade edges between adjacent agents with the objective of deterring the influence of external agents attached to the network, feeding in a constant mean signal. We consider a scenario where agents connected to R broadcast acknowledgment signals informing the network that they are being unfavorably influenced and so all agents within the graph are aware of the local directions of the external agents and more specifically their neighboring agents that are closer to the external agents. We denote these agents in the set $\mathcal{I}(v_i)$ for agent v_i and define if formally as the set composed of all agents that are neighbors of v_i and lie on the shortest path between v_i and any $r_j \in R$.

Protocol 1 Increased mean resilience edge swap

foreach *Agent* v_i **do**
  if $\exists v_j, v_k \in \mathcal{N}(v_i)$, $v_j \neq v_k$ *and* $v_j, v_k \notin \mathcal{I}(v_i)$ **then**
    $E \to E - \{v_i,v_j\} + \{v_j,v_k\}$
  end
end

We clarify that R is solely composed of unfriendly agents. The following lemma can be executed concurrently, in a random agent order, guarantees that $J^{\text{avg}}(\mathcal{T},\mathcal{R})$ increases, and a connected tree is maintained at each iteration. We denote edge removal and addition by the set notation "$-/+$."

Lemma 7: [Edge swap for improved mean resilience] Under **Protocol 1**, $J^{\text{avg}}(\mathcal{T},\mathcal{R})$ is strictly increasing.

Proof: If $v_m \in \mathcal{M}$ then for all $v_l \in \mathcal{N}(v_m)$ we have $v_m \in \mathcal{I}(v_l)$. Therefore in regard to **Protocol 1** $v_j, v_k \notin \mathcal{M}$. Then from Lemma 3 before the edge swap we have $E_{\text{eff}}(v_j) = E_{\text{eff}}(v_k) = E_{\text{eff}}(v_i) + 1$, after the edge swap $E_{\text{eff}}(v_j) = E_{\text{eff}}(v_i) + 2$ and all other agent's effective resistance increases by 1 or stays constant. Therefore, $J^{\text{avg}}(\mathcal{T},\mathcal{R})$ increases. ■

Under the decentralized unfriendly **Protocol 1**, for all single-external agent trees the graph will eventually reach the greatest $J^{\text{avg}}\left(\mathcal{T},\mathcal{R}^1\right) = (n+1)/2$ corresponding to a path graph with the external agent at an end. All other graphs will acquire a path-like appearance with the main path unaffected by the protocol's edge swaps.

The protocol was applied to a random tree graph on 40 agents with a single external agent connected to v_1. The path graph with the external agent attached to an end node was achieved after 100 edge swaps. A sample of the intermediate graphs is displayed in Figure 2. The metric $J^{\text{avg}}\left(\mathcal{T},\mathcal{R}^1\right)$ increased for each edge swap and no more edge swaps were possible when the tree became a path graph with $J^{\text{avg}}\left(\mathcal{T},\mathcal{R}^1\right) = 20.5$.

A complementary friendly protocol that aims to decrease $J^{\text{avg}}(\mathcal{T},\mathcal{R})$ can also be obtained from Lemma 7 [9].

Remark 8: With only local knowledge, i.e., $\mathcal{I}(v_i)$ and $\mathcal{N}(v_i)$, Lemma 7 describes the *only* edge swaps available to v_i that guarantee J^{avg} increases [9].

A by-product of this remark is that a strictly increasing local-knowledge protocol cannot guarantee the tree graph with the largest J^{avg} for $r > 1$ external agents.

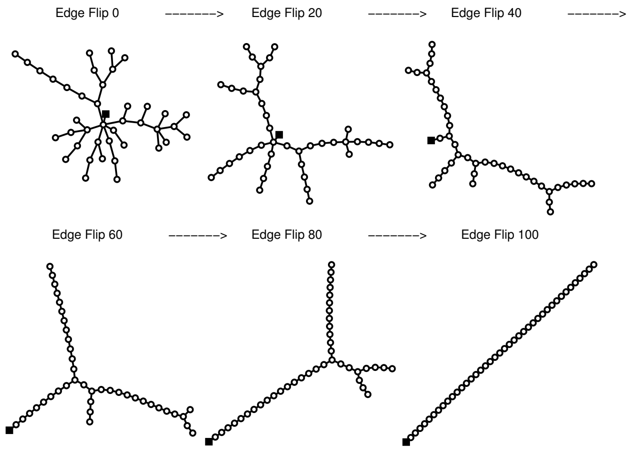


Figure 2. Selected iterations of a random tree graph with an external agent attached (square) running **Protocol 1**.

IV. Variance resilience

It is not uncommon that the mean is not of central interest and that adjustment of the variance of the states may be more desirable. With this in mind, the controllability gramian, defined as $P =: \int_0^\infty e^{A\tau} BB^T e^{A^T\tau} d\tau$, proves to be particularly suitable for such an analysis. We will focus on $\mathbf{tr}(P)$, as the average variance is $1/n \sum_{i=1}^n \mathbb{E}\left(\tilde{x}_i^2(t)\right) = (1/n)\,\mathrm{tr}\left(\mathbb{E}\left[\tilde{x}(t)\tilde{x}^T(t)\right]\right) = (1/n)\,\mathbf{tr}(P)$ as $t \to \infty$ over n outputs due to a white noise input with covariance I.

We note that P will be dependent on $\mathcal{G}$ and $\mathcal{R}$ and so henceforth is denoted by $P(\mathcal{G},\mathcal{R})$. The variance resilience is a metric quantifying the network's susceptibility to white noise from external agents.

Definition 9: The *variance resilience* of a network is the trace of the controllability gramian $\mathbf{tr}(P(\mathcal{G},\mathcal{R}))$.

The following section will provide more insights into the variance resilience.

A. Analysis of Variance resilience

Directly from the definition of the controllability gramian one has

$$\mathbf{tr}(P(\mathcal{G},\mathcal{R})) = \mathbf{tr}(\int_0^\infty e^{A(\mathcal{G},\mathcal{R})\tau} B\left(\mathcal{R}\right) B\left(\mathcal{R}\right)^T e^{A(\mathcal{G},\mathcal{R})^T\tau} d\tau)$$

$$= \mathbf{tr}(M\left(\mathcal{R}\right) \int_0^\infty e^{2A(\mathcal{G},\mathcal{R})\tau} d\tau) = -\frac{1}{2}\mathbf{tr}\left(M\left(\mathcal{R}\right) A(\mathcal{G},\mathcal{R})^{-1}\right).$$

Lemma 10: [General variance resilience] For a connected graph $\mathcal{G}$ the variance resilience is

$$\mathbf{tr}(P(\mathcal{G},\mathcal{R})) = \frac{1}{2}\sum_{v_i \in \pi(\mathcal{E}_R)} E_{\text{eff}}\left(v_i\right).$$

Proof: We note that $M\left(\mathcal{R}\right)$ is a purely diagonal matrix with $\left[M\left(\mathcal{R}\right)\right]_{ii} = 1$ if $v_i \in \pi\left(\mathcal{E}_R\right)$ and $\left[M\left(\mathcal{R}\right)\right]_{ii} = 0$, otherwise. Therefore $\left[M\left(\mathcal{R}\right) A(\mathcal{G},\mathcal{R})^{-1}\right]_{ii} = \left[A(\mathcal{G},\mathcal{R})^{-1}\right]_{ii}$ if $v_i \in \pi\left(\mathcal{E}_R\right)$ and $\left[M\left(\mathcal{R}\right) A(\mathcal{G},\mathcal{R})^{-1}\right]_{ii} = 0$, otherwise. The statement of the lemma now follows. ■

Corollary 11: [Single-external variance resilience] For a connected graph and the influence model (1) with one external agent,

$$\mathbf{tr}(P(\mathcal{G},\mathcal{R})) = 1/2.$$

Proof: The effective resistance of $\{v_i, r_1\} = \mathcal{E}_R$ is $E_{\text{eff}}(v_i) = 1$ as there is only one resistor link between v_i and r_1. The corollary follows. ■

The implication of Corollary 11 is that on average, a single-external agent attached to an n-agent connected graph has the same reduction in average variance to white noise and energy dissipation from an impulse input regardless of the structure of the network and where the external agent is connected.

Proposition 12: [Multiple-external variance resilience] For connected graphs and the influence model (1) with r external agents, the variance damping measure is bounded below by a graph with $\mathcal{M} = \pi\left(\mathcal{E}_R\right)$ in which case $\mathbf{tr}(P(\mathcal{G},\mathcal{R})) \geq \frac{r}{2\sqrt{5}}$.

Proof: By Rayleigh's Monotonicity Principle[2] the minimum effective resistance will occur when the main path is only composed of the r agents $\pi\left(\mathcal{E}_R\right)$. Of these r agent subgraphs, the path graph with the most resistors in parallel will have the smallest effective resistance and therefore the smallest value of $\mathbf{tr}(P(\mathcal{G},\mathcal{R}))$. The eigenvalues of the Laplacian of a r-node path graph are $\lambda_{r+1-i}\left(\mathcal{P}\right) = 2{+}2\cos\frac{\pi i}{r}$, for $i = 1,\ldots,r$ [11]. For $\tilde{\mathcal{R}}$ corresponding to an external agent attached to every agent in $\mathcal{P}$, from (2) and $M(\tilde{\mathcal{R}}) = I$ then $\lambda_{r+1-i}(-A(\mathcal{P},\tilde{\mathcal{R}})) = \lambda_{r+1-i}\left(\mathcal{P}\right){+}1$. Thus, $\mathbf{tr}(P(\mathcal{G},\tilde{\mathcal{R}})) = \frac{1}{2}\sum_{i=1}^r (3 + 2\cos\frac{\pi i}{r})^{-1} \geq \frac{r}{2\sqrt{5}}$. ■

B. Adaptive Protocol to Degrade the Variance Resilience for Trees

We propose another protocol for tree graphs $\mathcal{T}$ now with the objective of reducing the state variance due to external agents attached to the network and feeding in white noise with covariance I, i.e., decreasing the variance resilience. Again the protocol involves local edge trades executed concurrently, in a random agent order, guarantees that $\mathbf{tr}(P(\mathcal{T},\mathcal{R}))$ decreases and a connected tree is maintained at each iteration.

We note for a connected tree graph $\mathbf{tr}(P(\mathcal{T},\mathcal{R}))$ is only dependent on $d\left(r_i, r_j\right)$ for all $\{r_i, r_j\}$ pairs in R, and so only dependent on the graph of the main path with agents $\mathcal{M}$.

Lemma 13: [Edge Swap for decreased variance resilience] Under **Protocol 2**, $\mathbf{tr}\left(P(\mathcal{T},\mathcal{R})\right)$ is decreasing.

Proof: Firstly, as v_i has $|\mathcal{I}\left(v_i\right)| = 2$ then $v_i \in \mathcal{M}$. As v_j and v_k are closer to an external agent than the main path agent v_i then $v_j, v_k \in \mathcal{M}$. The edge swap involves removing v_i from $\mathcal{M}$. On the other hand, $\mathbf{tr}(P(\mathcal{T},\mathcal{R}))$ only depends on the subgraph composed of agents in $\mathcal{M}$, so the effect is to reduce an edge resistance within the electrical network representing this subgraph. By Rayleigh's Monotonicity Law, $\sum_{\{v_i,r_j\}\in\mathcal{E}_R} E_{\text{eff}}\left(v_i\right)$ will not increase and the lemma follows. ■

Single-external agent trees will remain unaffected by **Protocol 2**. For double-external agent trees the main path will degenerate to $\{v_i, v_j\} = \mathcal{M}$ where $\left(\{v_i, r_1\}, \{v_j, r_2\}\right) = \mathcal{E}_R$. The protocol was run on a 40 agent random tree with 3

[2]Rayleigh's Monotonicity Law states that if the edge resistance in an electrical network is decreased, then the effective resistance between any two agents in the network can only decrease [10].

Protocol 2 Decreased variance resilience edge swap

foreach *Agent* $v_i \notin \pi(\mathcal{E}_R)$ **do**
  if $\exists v_j, v_k \in \mathcal{N}(v_i)$*,* $v_j \neq v_k$ *and* $v_j, v_k = \mathcal{I}(v_i)$ **then**
    $E \to E - \{v_i, v_j\} + \{v_j, v_k\}$
  end
end

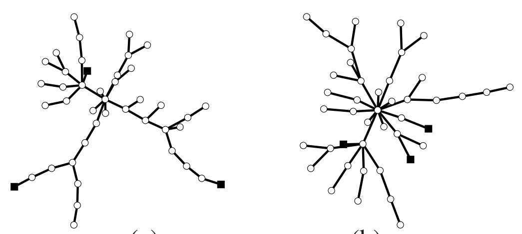


Figure 3. (a) Original random tree and (b) final graph with three external agents attached (squares) after applying **Protocol 2**.

external agents; the original and final graphs are displayed in Figure 3.

A complementary energy amplification protocol, that aims to increase **tr**$(P(\mathcal{T},\mathcal{R}))$, can also be obtained from Lemma 7 [9]. This protocol is suitable for impulse detection as larger **tr**$(P(\mathcal{T},\mathcal{R}))$ produces higher output energy $\int_0^\infty x(t)^T x(t)dt$.

Remark 14: The proposed protocol is the best local-knowledge, i.e., $\mathcal{I}(v_i)$ and $\mathcal{N}(v_i)$, edge swapping method for v_i and no guarantees can be made that the local-knowledge method will converge to the global-knowledge edge swap solution [9].

We previously remarked that **Protocol 1** does not alter the main path. Consequently, by Lemma 10, the **tr**$(P(\mathcal{T},\mathcal{R}))$ is conserved throughout this protocol so that, although the mean resilience is altered, the variance resilience remains the same. The converse is not true as **Protocol 2** involve manipulations of the main path and, as remarked in the previous section, this can arbitrarily vary $J^{\text{avg}}(\mathcal{T},\mathcal{R})$. Generally speaking as **tr**$(P(\mathcal{T},\mathcal{R}))$ decreases under **Protocol 2** the graphs compress and so $J^{\text{avg}}(\mathcal{T},\mathcal{R})$ tends to decrease.

V. Game Theoretic Adaptive Protocol

The protocols proposed so far possess guarantees on increasing (or decreasing) the mean (or variance) resilience of the graph per edge swap. The weakness of these protocols is they tend to converge to graphs associated with a local minimum (or maximum) $J^{\text{avg}}(\mathcal{T},\mathcal{R})$ (or **tr**$(P(\mathcal{T},\mathcal{R}))$) with potentially sub-optimal performance. Furthermore the protocols cannot be applied concurrently, e.g., one that aims for high $J^{\text{avg}}(\mathcal{T},\mathcal{R})$ and low **tr**$(P(\mathcal{T},\mathcal{R}))$. We now present a protocol that exhibits these attributes, i.e., the final graphs are within guaranteed bounds of the optimal over all graphs for maximizing $J^{\text{avg}}(\mathcal{T},\mathcal{R})$ and minimizing **tr**$(P(\mathcal{T},\mathcal{R}))$, respectively, but no longer possess strictly increasing $J^{\text{avg}}(\mathcal{T},\mathcal{R})$ and decreasing **tr**$(P(\mathcal{T},\mathcal{R}))$. We will present the protocol and use game theoretic techniques to bound the protocol's performance.

In the following, the objective is to increase $J^{\text{avg}}(\mathcal{T},\mathcal{R})$ and **tr**$(P(\mathcal{T},\mathcal{R}))^{-1}$. This produces a graph that is a balance

Protocol 3 Increased mean resilience and decreased variance resilience edge swap

foreach *Agent* v_i **do**
  if $\exists v_j, v_k \in \mathcal{N}(v_i)$ *and* $v_j \neq v_k$*, and* $(v_j, v_k \notin \mathcal{I}(v_i))$ *or* $(v_i \notin \pi_2(\mathcal{E}_R)$ *and* $v_j, v_k \in \mathcal{I}(v_i))$ **then**
    $E \to E - \{v_i, v_j\} + \{v_j, v_k\}$
  end
end

between dampening the external agents' effect on the system's state mean and the state variance. **Protocol 3** concurrently applies **Protocol 1** and **2** with a slight adaption to the latter, specifically, relaxing v_j,$v_k = \mathcal{I}(v_i)$ to v_j,$v_k \in \mathcal{I}(v_i)$. The adaption guarantees that the main path subgraph will converge to a graph where $\mathcal{M} = \pi(\mathcal{E}_R)$. The remaining nodes in the graph will form paths connected to agents in $\pi(\mathcal{E}_R)$. There are many equilibria graphs that satisfy these properties. The specific equilibrium will depend on the initial graph structure and the sequence of edge swaps. The analysis of this protocol falls under a special class of game called a *potential game* [12], which exhibits certain guarantees which will be explored further.

A. Game theoretic Analysis

Game theory supplies tools to quantify a protocol's success where more than one final equilibria could be reached. Two metrics are generally used; the price of stability which is the ratio between the best acquirable and the optimal equilibria, and the price of anarchy which is the ratio of the worst acquirable and the optimal equilibria.

First we need to establish that the protocol indeed converges to some equilibrium; for this task we use the concept of a potential game. A potential function Φ is a function that maps a strategy vector (a vector of each agent's edge swap) $S = (S_1, S_2, \ldots, S_n)$ to some real valued number. The implementation of a strategy on graph $\mathcal{T}$ will alter it to produce a graph $\mathcal{T}(S)$. If a protocol is a potential game then: if $S_i' \neq S_i$ is an alternate strategy (edge swap) for agent i then the local cost benefit to the agent $u_i(S') - u_i(S)$ will mirror the change in the potential, i.e., condition sgn$(\Phi(S) - \Phi(S'))$ = sgn$(u_i(S') - u_i(S))$[3]. Consider the potential function $\Phi(\mathcal{T}(S),\mathcal{R}) = -\sum_{i=1}^n d(v_i, \Gamma(v_i))$, where $\Gamma(v_i)$ is defined in Section III-A. Therefore if the local cost of agent v_i is $u_i(\mathcal{T}(S),\mathcal{R}) = d(v_i,\Gamma(v_i))$ then this mirroring condition is met. **Protocol 3** satisfies this potential and local cost function, and so is a potential game.[4] Therefore **Protocol 3**, will always converge to an equilibrium [12].

We can now find the price of stability and anarchy for $J^{\text{avg}}(\mathcal{T},\mathcal{R})$ and **tr**$(P(\mathcal{T},\mathcal{R}))^{-1}$ under **Protocol 3**.

Proposition 15: [Price for mean resilience] Under **Protocol 3** for $J^{\text{avg}}(\mathcal{T},\mathcal{R})$ the price of stability is equal to 1 and the price of anarchy is less than or equal to r.

Proof: From Proposition 6 the optimal $J^{\text{avg}}(\mathcal{T},\mathcal{R})$ equilibrium is bounded as $\max_{(\mathcal{T},\mathcal{R})} J^{\text{avg}}(\mathcal{T},\mathcal{R}) \leq$

[3]The signum function is represented by sgn$(\cdot)$.

[4]This approach is similar to other network game problems [12].

$\frac{1}{2n}\left((n-r)^2+3(n-r)+r+\frac{2}{r+1}\right)$. As the optimal $J^{\text{avg}}(\mathcal{T},\mathcal{R})$ equilibrium (Proposition 6) is acquirable under **Protocol 3**, the price of stability is equal to 1. The worst case $J^{\text{avg}}(\mathcal{T},\mathcal{R})$ equilibrium will correspond to network with the main-path subgraph a path $\mathcal{P}$ (by Proposition 12) with $\frac{r}{\sqrt{5}} \geq \sum_{v_i\in\mathcal{M}} E_{\text{eff}}(v_i)$ and as each of these agents has $E_{\text{eff}}(v_i) \leq 1$, the worst case graph will have $\lfloor (n-r)/r \rfloor = (n-r)/r$ agents attached as a path to each of the main-path agents. Applying Lemma 3, $\min_{(\mathcal{T},\mathcal{R})} J^{\text{avg}}(\mathcal{T},\mathcal{R}) > \frac{1}{n}[r/\sqrt{5}+r\sum_{i=1}^{(n-r)/r}(1/\sqrt{5}+i)]$ $= \frac{1}{2nr}((n-r)^2+\left(\frac{2}{\sqrt{5}}+1\right)r(n-r)+\frac{2}{\sqrt{5}}r^2)$
For $r=1$, the protocol always acquires the optimal equilibrium of a path graph with an external agent connected to an end node so the price of anarchy is 1. For $1<r\leq n$,

$$\text{Price of anarchy} = \max_{(\mathcal{T},\mathcal{R})} J^{\text{avg}}(\mathcal{T},\mathcal{R}) / \min_{(\mathcal{T},\mathcal{R})} J^{\text{avg}}(\mathcal{T},\mathcal{R})$$
$$\leq r\frac{(n-r)^2+3(n-r)+r+\frac{2}{r+1}}{(n-r)^2+\left(\frac{2}{\sqrt{5}}+1\right)r(n-r)+\frac{2}{\sqrt{5}}r^2} < r,$$

thus proving the proposition. ∎

Proposition 16: [Price for variance resilience] Under **Protocol 3**, for $\mathbf{tr}(P(\mathcal{T},\mathcal{R}))$ the price of stability is equal to 1 and the price of anarchy is less than $11\sqrt{5}/20 \approx 1.23$.

Proof: From Proposition 12, the optimal $\mathbf{tr}(P(\mathcal{T},\mathcal{R}))$ equilibrium corresponds to a path $\mathcal{P}$ main-path subgraph with $r/2\sqrt{5} < \mathbf{tr}(P(\mathcal{P},\mathcal{R}))$. As the optimal $\mathbf{tr}(P(\mathcal{T},\mathcal{R}))$ equilibrium is acquirable under **Protocol 3** (by Proposition 12), the price of stability is equal to 1. The optimal is guaranteed for $r=1,2,3$ (main path subgraph of a path) and so the price of anarchy is 1 for $r\leq 3$. From Proposition 6, the worst acquirable $\mathbf{tr}(P(\mathcal{T},\mathcal{R}))$ equilibrium is associated with a star $\mathcal{S}$ main-path subgraph with $\mathbf{tr}(P(\mathcal{S},\mathcal{R})) = \left(r^2+r+2\right)/4(r+1)$. For $3<r\leq n$, the Price of anarchy$= \min_{(\mathcal{T},\mathcal{R})}\mathbf{tr}(P(\mathcal{T},\mathcal{R}))/\max_{(\mathcal{T},\mathcal{R})}\mathbf{tr}(P(\mathcal{T},\mathcal{R})) < \sqrt{5}(r^2+r+2)/2\left(r^2+r\right) < 11\sqrt{5}/20$, thus proving the proposition. ∎

B. Protocol Comparison

Protocol 3 was applied to a 40 node tree graph with 7 external agents. For comparison, **Protocol 1** (increasing mean resilience) and **Protocol 2** (decreasing variance resilience) were applied to the same graph. The original and final graphs for each protocol appear in Figure 4 while the metrics $J^{\text{avg}}(\mathcal{T},\mathcal{R})$ and $\mathbf{tr}(P(\mathcal{T},\mathcal{R}))$ for each compared to the optimal tree graphs for $J^{\text{avg}}(\mathcal{T},\mathcal{R})$ and $\mathbf{tr}(P(\mathcal{T},\mathcal{R}))$ are displayed in Figure 5.

We note that **Protocol 3** outperformed **Protocols 1** and **2**. The ratio of the optimal to the final equilibrium under **Protocol 3** was within 1.6 for $J^{\text{avg}}(\mathcal{T},\mathcal{R})$ and within 1.1 for $\mathbf{tr}(P(\mathcal{T},\mathcal{R}))^{-1}$, agreeing with the game-theoretic bounds found in Propositions 15 and 16.

VI. Conclusion

This paper presents a system-theoretic approach to the notion of semi-autonomy. Metrics were introduced and analyzed that quantify the network's ability, via its topology, to resist the influence of external agents. Decentralized protocols involving adapting the network structure were proposed for tree graphs to vary these metrics. Finally the protocols were extended to a hybrid protocol and analyzed using game theoretic techniques.

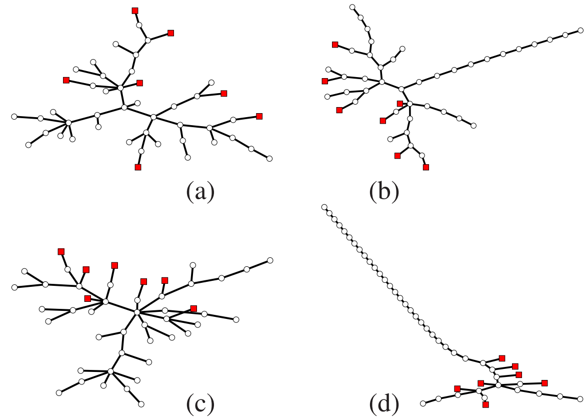


Figure 4. (a) Original random tree and final graph with seven external agents attached (squares) after applying (b) **Protocol 1**, (c) **Protocol 2** and (d) **Protocol 3**.

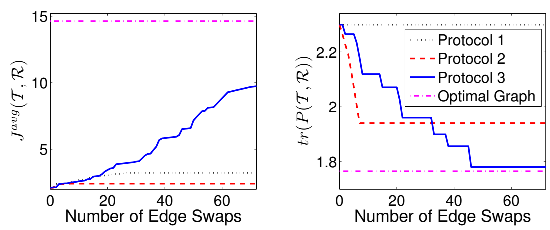


Figure 5. $J^{\text{avg}}(\mathcal{T},\mathcal{R})$ and $\mathbf{tr}(P(\mathcal{T},\mathcal{R}))$ after each edge swap from **Protocols 1**, **2** and **3** applied to the original graph in Figure 4 as well as the respective optimal tree graphs with 40 nodes and 7 external agents.

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