Semi-Autonomous Networks:
Network Resilience and Adaptive Trees
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Abstract—This paper examines the dynamics of a networked multi-agent system operating under a consensus-type coordination algorithm that can be influenced by external agents. We refer to this class of networks as semi-autonomous. Within such a class, we consider a network’s resilience to the influence of external agents delivering a test signal, namely a Gaussian noise with a given mean. Specifically, we examine the resultant mean and variance of the states of the agents in the network, via metrics dubbed as mean and variance resilience, as well as relate these quantities to circuit-theoretic notions of the network. These metrics are then used to propose adaptive protocols for tree graphs to increase or decrease the mean and variance resilience. Finally, a hybrid protocol is proposed which is shown to have a guaranteed performance using game-theoretic techniques. All protocols involve decentralized edge swaps that can be performed in parallel, asynchronously, and require only local agent information of the graph structure.

Index Terms—Semi-autonomous networks; Consensus protocol; Effective Resistance; Coordinated control over networks

I. INTRODUCTION

Consensus-type algorithms provide effective means for distributed information-sharing and control for networked, multi-agent systems in settings such as multi-vehicle control, formation control, swarming, and distributed estimation; see for example, [1], [2], [3], [4]. An appeal of consensus algorithms is their ability to operate autonomously over simple trusting agents. This has the added benefit that external (control) agents, perceived as native agents, can seamlessly attach to the network and steer it in particular directions. These additional agents, ignoring consensus rules, will influence the system dynamics compared to the unforced networked system resulting in scenarios such as leader-follower [2] and drift correction [5]. The detriment is that this same approach can be adopted by malicious infiltrating agents. We refer to this class of systems, with friendly and/or unfriendly attached nodes, as semi-autonomous networks.

In this paper we examine the resilience of a network to the effect of an external agent injecting a test signal, namely a white Gaussian signal, into the network. The resilience of the network is measured in terms of the mean and variance of the agents’ state; we refer to these metrics as the mean and variance resilience. For the mean resilience, an electrical network analogy is used to measure the average cost of convergence. For the variance resilience, on the other hand, we use the controllability gramian of the resulting system.

Both types of resilience are subsequently used to propose decentralized protocols for tree graphs that adaptively aim to improve or degrade the network’s resilience by performing local edge swaps.

The contribution of the paper is two fold. First, we provide a system-theoretic approach to examine and reason about the resilience of influenced coordination protocols operating over a graph. Secondly, we provide distributed adaptation schemes that the agents in the network can adopt in order to alter the influence of the external agents on the network operation.

II. BACKGROUND AND MODEL

We provide a brief background on constructs and models that will be used in this paper.

An undirected graph $\mathcal{G} = (V, E)$ is defined by a node set $V$ with cardinality $n$ and an edge set $E$ comprised of pairs of nodes, where nodes $v_i$ and $v_j$ are adjacent if $(v_i, v_j) \in E$. We denote the set of nodes adjacent to $v_i$ as $\mathcal{N}(v_i)$ and the minimum path length, induced by the graph, between nodes $v_i$ and $v_j$ as $d(v_i, v_j)$. The degree $\delta_i$ of node $v_i$ is the number of its adjacent nodes. The degree matrix $\Delta(\mathcal{G}) \in \mathbb{R}^{n \times n}$ is a diagonal matrix with $\delta_i$ at position $(i, i)$. The adjacency matrix is an $n \times n$ symmetric matrix with $[A(\mathcal{G})]_{ij} = 1$ when $(v_i, v_j) \in E$ and $[A(\mathcal{G})]_{ij} = 0$ otherwise. The combinatorial Laplacian is defined as $L(\mathcal{G}) = \Delta(\mathcal{G}) - A(\mathcal{G})$.

Now consider $x_i(t) \in \mathbb{R}$ to be node (or for our case agent) $v_i$’s state at time $t$. The continuous-time consensus protocol is defined as $\dot{x}_i(t) = \sum_{\{i, j\} \in E} (x_j(t) - x_i(t))$ where agent pairs $\{i, j\} \in E$ are able to communicate. In a compact form with $x(t) \in \mathbb{R}^n$, the collective dynamics is represented as $\dot{x}(t) = -L(\mathcal{G})x(t)$ with $L(\mathcal{G})$ being the Laplacian of the underlying interaction topology [1].

We next introduce a model of influenced consensus associated with a control pair $\mathcal{R} = (R, \mathcal{E}_R)$, where $R \in \mathbb{R}^r$ is the set of $r$ external agent and $\mathcal{E}_R \subseteq R \times V$ is the set of edges used by the external agents to inject signals into the network. It is assumed that each external agent $r_j \in R$ is attached to exactly one node $v_i \in V$ along the edge $\{r_j, v_i\} \in \mathcal{E}_R \subseteq \mathbb{R}^r$ and subsequently delivers a signal $u_j(t) \in \mathbb{R}$.

The resulting influenced system now assumes the form, 

$$\dot{x}(t) = A(\mathcal{G}, \mathcal{R})x(t) + B(\mathcal{R})u(t),$$ 

where $B(\mathcal{R}) \in \mathbb{R}^{n \times r}$ with $[B(\mathcal{R})]_{ij} = 1$ when $\{r_j, v_i\} \in \mathcal{E}_R$ and $[B(\mathcal{R})]_{ij} = 0$ otherwise, and

$$A(\mathcal{G}, \mathcal{R}) = - (L(\mathcal{G}) + M(\mathcal{R})) \in \mathbb{R}^{n \times n},$$

where $M(\mathcal{R}) = B(\mathcal{R})B(\mathcal{R})^T \in \mathbb{R}^{n \times n}$. We also define the special type of single-agent control as $\mathcal{R}^s$ where $\mathcal{R} = \{r_1\}$.

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The matrix $A(G, R)$ is negative definite (and so invertible) if the graph is connected. In order to quantify the resilience of a network to resist the influence of external agents, the following two observations are in order: (1) the dynamics of the mean of agents’ state is captured by model (1) where $u$ is replaced by the mean of the Gaussian noise $u_c$, (2) when the underlying graph is connected, all agents’ state converge in the mean to $u_c$. The last statement is a direct consequence of Proposition 1. We approach the network resilience problem from two fronts; first as a network’s ability to resist its agent states’ convergence to $u_c$ - dubbed mean resilience (Section III) - and secondly its ability to resist its agent states’ variance increasing due to a noisy external agent’s signal - dubbed variance resilience (Section IV).

### III. MEAN RESILIENCE

The mean resilience is a metric for the effectiveness of a network, via its topology, to resist convergence to the mean of external agents’ signal $u_c$. We derive the mean resilience as the cost incurred by external agents to steer the mean of the states to $u_c$. More specifically, noting that $1 = [1, \ldots, 1]^T$, $1_x \in \mathbb{R}^n$, $1_a \in \mathbb{R}^n$ and $A(G, R)^{-1} B 1_a = -1_x$, the convergence cost, with coordinate transform $\hat{x}(t) = x(t) - u_c 1_x$ can be derived as,

$$2 \int_0^T \tilde{x}^T \tilde{x} dt = \int_0^T \tilde{x}^T \tilde{x} + x^T \tilde{x} - x^T 1_x u_c - u_c^T 1_x \tilde{x} dt$$

$$= \int_0^T \tilde{x}^T x + x^T \tilde{x} + x^T A^{-1} B 1_a u_c + u_c 1_x^T B^T A^{-1} \tilde{x} dt$$

$$= \int_0^T (A x + B 1_a u_c)^T A^{-1} x + x^T A^{-1} (A x + B 1_a u_c) dt$$

$$= \int_0^T \tilde{x}^T A^{-1} \tilde{x} + \tilde{x}^T A^{-1} \tilde{x} dt = \int_0^T \frac{d}{dt} \tilde{x}^T A^{-1} \tilde{x} dt$$

$$= \tilde{x}(T)^T A^{-1} \tilde{x}(T) - \tilde{x}(0)^T A^{-1} \tilde{x}(0).$$

In order to parametrize the resilience of the network for a specific control set $R$, let us define the accumulative state mean over a finite time horizon $T$ and $\tilde{x}(0)$ uniformly distributed about the unit circle as

$$J_{\text{avg}}(G, R, T) = \mathbb{E}_{\tilde{x}(0) \sim U(0)} \left( 2 \int_0^T \tilde{x}(t)^T \tilde{x}(t) dt \right)$$

$$= \mathbb{E}_{\tilde{x}(0) \sim U(0)} \left( \tilde{x}(T)^T A^{-1} \tilde{x}(T) - \tilde{x}(0)^T A^{-1} \tilde{x}(0) \right)$$

$$= \mathbb{E}_{\tilde{x}(0) \sim U(0)} \text{tr} \left( \tilde{x}(0)^T \tilde{x}(0) \left( e^{AT} \right)^T A^{-1} e^{AT} - A^{-1} \right)$$

$$= \mathbb{E}_{\tilde{x}(0) \sim U(0)} \text{tr} \left( \tilde{x}(0)^T \tilde{x}(0) \left( e^{AT} \right)^T A^{-1} e^{AT} - A^{-1} \right)$$

$$= \frac{1}{n} \text{tr} \left( \left( e^{AT} I - A^{-1} \right) A^{-1} \right) = \frac{1}{n} \sum_{i=1}^n \frac{1}{n} \left( 1 - e^{-2 \lambda_i (-A)^T} \right).$$

We assume time $T$ is unknown but sufficiently large, justifying the use of $J_{\text{avg}}(G, R, \infty)$ as a measure of mean resilience. In fact for brevity, we let $J_{\text{avg}}(G, R, \infty) = J_{\text{avg}}(G, R)$. We can now formally define our metric.

**Definition 2:** The mean resilience of a network is the average cost incurred by external agents to steer the mean state of the entire network to its own mean value, over an infinite horizon, and is equal to

$$J_{\text{avg}}(G, R, \infty) = \frac{1}{n} \text{tr} \left( -A(G, R)^{-1} \right).$$

The following section will provide more insight into the mean resilience.

### A. Analysis of Mean resilience

It has previously been established that the diagonal of $-A(G, R)^{-1}$ has a resistive electrical network interpretation [6]. In this setup, the agents $V$ and $R$, defined in Section II, represent connection points between 1 ohm resistors corresponding to the communication edges $E$ and $E_R$. In addition, all connection points corresponding to $R$ are electrically shorted together. The effective resistance between two connection points in an electrical network is defined as the potential drop between the two points, when a 1 Amp current source is connected across the two points. The $i$-th diagonal element of $-A(G, R)^{-1}$ is the effective resistance $\mathcal{E}_{\text{eff}}(v_i)$ between the common shorted external agents $R$ and $v_i$. An example of the equivalent electrical network is displayed in Figure 1. The implication is that

$$J_{\text{avg}}(G, R, \infty) = \frac{1}{n} \sum_{i=1}^n \mathcal{E}_{\text{eff}}(v_i).$$

Tree graphs are often adopted for agent-to-agent communication topologies as they minimize edge (communication) costs while maintaining connectivity. We define some properties of $J_{\text{avg}}(G, R)$ specific to trees.

In this direction, let us first define the special set of agents that lie on any of the shortest paths between agents in $R$ as main path agents designated by set $\mathcal{M}$. This is a unique set for a given pair $(G, R)$, where there exists a unique agent $v_j \in \mathcal{M}$ that has a shorter minimum path to $v_i$ than any other agent in $\mathcal{M}$, we define this agent as $\Gamma(v_i)$, i.e., $\Gamma(v_i)$ is the closest agent to $v_i$ that is a member of the main path. Therefore for tree graphs we can state the following.
Lemma 3: [Mean resilience for trees] For the $n$-agent connected tree $T$, the mean resilience is $J_{\text{avg}}(T, R) = \frac{1}{n} \left( \sum_{v_i \in M} E_{\text{eff}}(v_i) + \sum_{v_i \in M} \left[ E_{\text{eff}}(\Gamma(v_i)) + d(v_i, \Gamma(v_i)) \right] \right)$.

Proof: If $v_i \notin M$ then the equivalent electrical network involving $v_i$ can be simplified into a resistor representing $E_{\text{eff}}(\Gamma(v_i))$ ohms in series with $d(v_i, \Gamma(v_i)) \times 1$ ohm resistors. The result then follows from (3).

There is an intuitive link between the centrality of an agent in a network and its influence on the network’s dynamics. This correlation becomes apparent for tree graphs in the following.

Corollary 4: [Single-external mean resilience] For the $n$-agent connected tree $T$ the mean resilience of the network to a single external agent attached to any agent $v_i \in V$ is $J_{\text{avg}}(T, R^i) = \frac{1}{n} \left( \sum_{j=1}^{n} d(v_i, v_j) + n \right)$.

Proof: Follows from Lemma 3 with $v_i = M$ and $E_{\text{eff}}(v_i) = 1$.

Corollary 5: [Single-external mean resilience bounds] For the $n$-agent connected tree $T$ the mean resilience of the network to a single external agent attached to any agent $v_i \in V$ is bounded as $2 - \frac{1}{n} \leq J_{\text{avg}}(T, R^i) \leq \frac{1}{2} (n + 1)$.

Proof: Over all trees, the central node of the star graph has the smallest accumulative distance of $n - 1$ to all other nodes and an end node of the path graph has the largest accumulative distance of $\sum_{i=1}^{n-1} i$ to all other nodes.

Proposition 6: [Multi-external mean resilience bounds] For the $n$-agent connected tree $T$ the mean resilience of $r$ external agents attached to any set of agents in $V$ is bounded above by a graph with all main path nodes satisfying $v_i \in \pi(E_R)$ and $J_{\text{avg}}(T, R) \leq \frac{1}{2n} \left( (n-r)^2 + 3(n-r) + r + 2/(r+1) \right)$.

Proof: From our effective resistance interpretation of $J_{\text{avg}}(T, R^i)$ (3), we note that adding resistors in series generates a higher resistance than adding in parallel. Therefore, $\text{argmax}_{\Gamma(v_i)} J_{\text{avg}}(T, R^i)$ implies $M = \pi(E_R)$. Furthermore from Lemma 3, the largest accumulative distance for $v_i \notin M$ will correspond to a path connected to the highest effective resistance node of $M$. Now the main path subgraph with the highest effective resistance sum is the star graph with the least number of parallel resistors. Applying resistor rules we find that the star graph $S$ with an external agent connected to each node, i.e., $r = n$ is the largest mean resilience graph with $J_{\text{avg}}(S, R) = (r^2 + r + 2)/2r(r+1)$. Similarly, the effective resistance of an agent in the main path subgraph is $E_{\text{eff}}(v_i) \leq 1$ as the equivalent electric network is a parallel resistor cascade of $1\Omega$ resistors. Combining this bound and the main path subgraph $S$, we have $J_{\text{avg}}(T, R) \leq \frac{1}{2n} \left( r J_{\text{avg}}(S, R) + \sum_{i=1}^{n-r} (E_{\text{eff}}(v_i) + i) \right) \leq \frac{1}{2n} \left( (n-r)^2 + 3(n-r) + r + 2/(r+1) \right)$.

B. Adaptive Protocol to Improve the Mean Resilience for Trees

We now can propose a protocol over a tree graph $T$ to locally trade edges between adjacent agents with the objective of deterring the influence of external agents attached to the network, feeding in a constant mean signal. We consider a scenario where agents connected to $R$ broadcast acknowledgment signals informing the network that they are being unfavorably influenced and so all agents within the graph are aware of the local directions of the external agents and more specifically their neighboring agents that are closer to the external agents. We denote these agents in the set $\mathcal{I}(v_i)$ for agent $v_i$ and define if formally as the set composed of all agents that are neighbors of $v_i$ and lie on the shortest path between $v_i$ and any $r_j \in R$.

We clarify that $R$ is solely composed of unfriendly agents. The following lemma can be executed concurrently, in a random agent order, guarantees that $J_{\text{avg}}(T, R)$ increases, and a connected tree is maintained at each iteration. We denote edge removal and addition by the set notation “$-/+$."

Lemma 7: [Edge swap for improved mean resilience] Under Protocol 1, $J_{\text{avg}}(T, R)$ is strictly increasing.

Proof: If $v_{m} \in M$ then for all $v_j \in N(v_{m})$ we have $v_{m} \notin \mathcal{I}(v_{j})$. Therefore in regard to Protocol 1 $v_j, v_k \notin M$. Then from Lemma 3 before the edge swap we have $E_{\text{eff}}(v_j) = E_{\text{eff}}(v_k) = E_{\text{eff}}(v_i) + 1$, after the edge swap $E_{\text{eff}}(v_j) = E_{\text{eff}}(v_i) + 2$ and all other agent’s effective resistance increases by 1 or stays constant. Therefore, $J_{\text{avg}}(T, R)$ increases.

Under the decentralized unfriendly Protocol 1, for all single-external agent trees the graph will eventually reach the greatest $J_{\text{avg}}(T, R^i) = (n+1)/2$ corresponding to a path graph with the external agent at an end. All other graphs will acquire a path-like appearance with the main path unaffected by the protocol’s edge swaps.

The protocol was applied to a random tree graph on 40 agents with a single external agent connected to $v_1$. The path graph with the external agent attached to an end node was achieved after 100 edge swaps. A sample of the intermediate graphs is displayed in Figure 2. The metric $J_{\text{avg}}(T, R^i)$ increased for each edge swap and no more edge swaps were possible when the tree became a path graph with $J_{\text{avg}}(T, R^i) = 20.5$.

A complementary friendly protocol that aims to decrease $J_{\text{avg}}(T, R)$ can also be obtained from Lemma 7 [9].

Remark 8: With only local knowledge, i.e., $\mathcal{I}(v_i)$ and $N(v_i)$. Lemma 7 describes the only edge swaps available to $v_i$ that guarantee $J_{\text{avg}}$ increases [9]. A by-product of this remark is that a strictly increasing local-knowledge protocol cannot guarantee the tree graph with the largest $J_{\text{avg}}$ for $r > 1$ external agents.
IV. VARIANCE RESILIENCE

It is not uncommon that the mean is not of central interest and that adjustment of the variance of the states may be more desirable. With this in mind, the controllability gramian, defined as $P = \int_0^\infty e^{A^T}B B^T e^{A T}\,d\tau$, proves to be particularly suitable for such an analysis. We will focus on $\text{tr}(P)$, as the average variance is $1/n \sum\limits_{i=1}^n \mathbb{E}(x_i^2(t)) = (1/n) \text{tr}(P)$ as $t \to \infty$ over $n$ outputs due to a white noise input with covariance $I$.

We note that $P$ will be dependent on $G$ and $R$ and so henceforth is denoted by $P(G, R)$. The variance resilience is a metric quantifying the network’s susceptibility to white noise from external agents.

Definition 9: The variance resilience of a network is the trace of the controllability gramian $\text{tr}(P(G, R))$.

The following section will provide more insights into the variance resilience.

A. Analysis of Variance Resilience

Directly from the definition of the controllability gramian one has

$$\text{tr}(P(G, R)) = \text{tr}\left(\int_0^\infty e^{A(G, R) T} B(R) B^T e^{A(G, R) T} \,d\tau\right)$$

$$= \text{tr}\left(M(R) \int_0^\infty e^{2A(G, R) r} \,d\tau \right) = -\frac{1}{2} \text{tr} \left(M(R) A(G, R)^{-1}\right).$$

Lemma 10: [General variance resilience] For a connected graph $G$ the variance resilience is

$$\text{tr}(P(G, R)) = \frac{1}{2} \sum_{v_i \in \pi(E_R)} E_{\text{eff}}(v_i).$$

Proof: We note that $M(R)$ is a purely diagonal matrix with $[M(R)]_{ii} = 1$ if $v_i \in \pi(E_R)$ and $[M(R)]_{ii} = 0$, otherwise. Therefore $[M(R) A(G, R)^{-1}]_{ii} = [A(G, R)^{-1}]_{ii}$ if $v_i \in \pi(E_R)$ and $[M(R) A(G, R)^{-1}]_{ii} = 0$, otherwise. The statement of the lemma now follows.

Corollary 11: [Single-external variance resilience] For a connected graph and the influence model (1) with one external agent,

$$\text{tr}(P(G, R)) = 1/2.$$
Protocol 2 Decreased variance resilience edge swap

foreach Agent $v_i \notin \pi(ER)$ do
  if $\exists v_j, v_k \in N(v_i), v_j \neq v_k$ and $v_j \neq v_k$ then
    $E \rightarrow E - \{v_i, v_j\} + \{v_j, v_k\}$
  end
end

Figure 3. (a) Original random tree and (b) final graph with three external agents attached (squares) after applying Protocol 2.

external agents; the original and final graphs are displayed in Figure 3.

A complementary energy amplification protocol, that aims to increase $tr(P(T, R))$, can also be obtained from Lemma 7 [9]. This protocol is suitable for impulse detection as larger $tr(P(T, R))$ produces higher output energy $\int_0^\infty x(t)^T x(t)dt$.

Remark 14: The proposed protocol is the best local-knowledge, i.e., $I(v_i)$ and $N(v_i)$, edge swapping method for $v_i$ and no guarantees can be made that the local-knowledge method will converge to the global-knowledge edge swap solution [9].

We previously remarked that Protocol 1 does not alter the main path. Consequently, by Lemma 10, the $tr(P(T, R))$ is conserved throughout this protocol so that, although the mean resilience is altered, the variance resilience remains the same. The converse is not true as Protocol 2 involve manipulations of the main path and, as remarked in the previous section, this can arbitrarily vary $J_{avg}^v(T, R)$. Generally speaking as $tr(P(T, R))$ decreases under Protocol 2 the graphs compress and so $J_{avg}^v(T, R)$ tends to decrease.

V. GAME THEORETIC ADAPTIVE PROTOCOL

The protocols proposed so far possess guarantees on increasing (or decreasing) the mean (or variance) resilience of the graph per edge swap. The weakness of these protocols is they tend to converge to graphs associated with a local minimum (or maximum) $J_{avg}^v(T, R)$ (or $tr(P(T, R))$) with potentially sub-optimal performance. Furthermore the protocols cannot be applied concurrently, e.g., one that aims for high $J_{avg}^v(T, R)$ and low $tr(P(T, R))$. Beyond this, we now present a protocol that exhibits these attributes, i.e., the final graphs are within guaranteed bounds of the optimal over all graphs for maximizing $J_{avg}^v(T, R)$ and minimizing $tr(P(T, R))$, respectively, but no longer possess strictly increasing $J_{avg}^v(T, R)$ and decreasing $tr(P(T, R))$. We will present the protocol and use game theoretic techniques to bound the protocol’s performance.

In the following, the objective is to increase $J_{avg}^v(T, R)$ and $tr(P(T, R))^{-1}$. This produces a graph that is a balance between dampening the external agents’ effect on the system’s state mean and the state variance. Protocol 3 concurrently applies Protocol 1 and 2 with a slight adaption to the latter, specifically, relaxing $v_j, v_k \in I(v_i)$ to $v_j, v_k \in I(v_i)$ or $(v_i \notin \pi(ER))$ and $(v_j, v_k \in I(v_i))$ then

$E \rightarrow E - \{v_i, v_j\} + \{v_j, v_k\}$

end

Protocol 3 Increased mean resilience and decreased variance resilience edge swap

foreach Agent $v_i$ do
  if $\exists v_j, v_k \in N(v_i)$ then
    $E \rightarrow E - \{v_i, v_j\} + \{v_j, v_k\}$
  end

end

A. Game theoretic Analysis

Game theory supplies tools to quantify a protocol’s success where more than one final equilibrium could be reached. Two metrics are generally used; the price of stability which is the ratio between the best acquirable and the optimal equilibria, and the price of anarchy which is the ratio of the worst acquirable and the optimal equilibria.

First we need to establish that the protocol indeed converges to some equilibrium; for this task we use the concept of a potential game. A potential function $\Phi$ is a function that maps a strategy vector (a vector of each agent’s edge swap) $S = (S_1, S_2, \ldots, S_n)$ to some real valued number. The implementation of a strategy on graph $T$ will alter it to produce a graph $T(S)$. If a protocol is a potential game then: if $S_i' \neq S_i$ is an alternate strategy (edge swap) for agent $i$ then the local cost benefit to the agent $u_i(S') - u_i(S)$ will mirror the change in the potential, i.e., condition $\text{sgn}(\Phi(S) - \Phi(S')) = \text{sgn}(u_i(S') - u_i(S))^3$. Consider the potential function $\Phi(S) = - \sum_i d(v_i, I(v_i))$, where $I(v_i)$ is defined in Section III-A. Therefore if the local cost of agent $v_i$ is $u_i(T(S), R) = d(v_i, I(v_i))$ then this mirroring condition is met. Protocol 3 satisfies this potential and local cost function, and so is a potential game. Therefore Protocol 3, will always converge to an equilibrium [12].

We can now find the price of stability and anarchy for $J_{avg}^v(T, R)$ and $tr(P(T, R))^{-1}$ under Protocol 3.

Proposition 15: [Price for mean resilience] Under Protocol 3 for $J_{avg}^v(T, R)$ the price of stability is equal to 1 and the price of anarchy is less than or equal to $r$.

Proof: From Proposition 6 the optimal $J_{avg}^v(T, R)$ equilibrium is bounded as $\max_{(T, R)} J_{avg}^v(T, R) \leq J_{opt}^v(T, R) \leq J_{opt}^v(T, R)$.
As the optimal $J_{\text{avg}}(T, R)$ equilibrium (Proposition 6) is acquirable under Protocol 3, the price of stability is equal to 1.

The worst case $J_{\text{avg}}(T, R)$ equilibrium will correspond to a path $P$ (by Proposition 12) with $E_{\text{eff}} \geq \sum_{v \in M} E_{\text{eff}}(v)$ and as each of these agents has $E_{\text{eff}}(v) \leq 1$, the worst case graph will have $(n - r)/r = (n - r)/r$ agents attached as a path to each of the main-path agents. Applying Lemma 3, $\min_{(T, R)} J_{\text{avg}}(T, R) > \frac{1}{2\sqrt{5}} r/\sqrt{5} + r/\sqrt{5} = \frac{1}{\sqrt{5}} (r^2 + 2) < r$.

Thus proving the proposition.

**Proposition 16:** [Price for variance resilience] Under Protocol 3, for $tr(P(T, R))$ the price of stability is equal to 1 and the price of anarchy is less than $11/\sqrt{5} = 2.3$.

**Proof:** From Proposition 12, the optimal $tr(P(T, R))$ equilibrium corresponds to a path $P$ main-path subgraph with $r/2\sqrt{5} < tr(P(P, R))$. As the optimal $tr(P(T, R))$ equilibrium is acquirable under Protocol 3 by Proposition 12, the price of stability is equal to 1. The optimal is guaranteed for $r = 1, 2, 3$ (main path subgraph of a path) and so the price of anarchy is 1 for $r \leq 3$. From Proposition 6, the worst acquirable $tr(P(T, R))$ equilibrium is associated with a star $S$ main-path subgraph with $tr(P(S, R)) = (r^2 + r + 2)/4(r + 1)$. For $3 < r \leq n$, the Price of anarchy $= \min_{(T, R)} tr(P(T, R)) / \max_{(T, R)} tr(P(T, R)) < \sqrt{5}(r^2 + 2)/2(r^2 + 1) < 11\sqrt{5}/20$, thus proving the proposition.

**B. Protocol Comparison**

Protocol 3 was applied to a 40 node tree graph with 7 external agents. For comparison, Protocol 1 (increasing mean resilience) and Protocol 2 (decreasing variance resilience) were applied to the same graph. The original and final graphs for each protocol appear in Figure 4 while the metrics $J_{\text{avg}}(T, R)$ and $tr(P(T, R))$ for each compared to the respective tree graphs are displayed in Figure 5.

We note that Protocol 3 outperformed Protocols 1 and 2. The ratio of the optimal to the final equilibrium under Protocol 3 was within 1.6 for $J_{\text{avg}}(T, R)$ and within 1.1 for $tr(P(T, R))^{-1}$, agreeing with the game-theoretic bounds found in Propositions 15 and 16.

**VI. CONCLUSION**

This paper presents a system-theoretic approach to the notion of semi-autonomy. Metrics were introduced and analyzed that quantify the network’s ability, via its topology, to resist the influence of external agents. Decentralized protocols involving adapting the network structure were proposed for tree graphs to vary these metrics. Finally the protocols were extended to a hybrid protocol and analyzed using game theoretic techniques.

**REFERENCES**


