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On the Controllability and Observability of Cartesian Product Networks

Airlie Chapman, Marzieh Nabi-Abdolyousefi, and Mehran Mesbahi

***Abstract*—The paper presents an analysis framework for a class of dynamic composite networks. These networks are formed from smaller factor networks via graph Cartesian products. We provide a composition method for extending the controllability and observability of the factor networks to that of the composite network. We then delve into the effectiveness of designing control and estimation algorithms for the composite network via symmetry in the network. Examples and applications are provided throughout the paper to demonstrate the results including a distributed output feedback stabilizer and a social network application.**

***Index Terms*—Network controllability; Network observability; Graph Cartesian product; Composite networks; Coordination algorithms**

I. Introduction

Complex dynamic networks are an integral part of the technological world including the internet, power grids, and communication networks, as well as in nature such as biological and chemical systems and social networks. An explosion of research in the area of network systems has resulted [2], [3], [4].

Network controllability and observability arises in situations where a networked system is influenced or observed by an external entity, such scenarios include networked robotic systems, human-swarm interaction, and network security [5], [6], as well as in areas such as quantum networks [7].

In this direction, Godsil [8] made an intriguing conjecture regarding the dynamics driven by the adjacency matrix of a graph. The conjecture states that the ratio of graphs that are uncontrollable, with a common input to all nodes, to the total number of graphs of the same order tends to zero as the order of the graph increases. On another front, extensive simulations have demonstrated that it is "unlikely" that single leader-follower Laplacian based consensus networks are controllable [9]. Together these observations imply that establishing network controllability is non-trivial and may be strongly dependent on the manifestation of the graph embedded in the dynamics.

Controllability for Laplacian networks has been established for special classes of graphs such as paths, circulants, grids, random and distance regular graphs [5], [10], [11], [12], [13], but to the authors' knowledge, no other large scale networks have been investigated. Liu *et al.* in [14] studied the structural controllability of complex networks.

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In this paper, we consider network controllability for a special class of graphs, namely large-scale networks which are Cartesian products of smaller factor-networks (factors). We present two control schemes for extending controllability of factors to the controllability of the composite network, dubbed the control product and layered control. The schemes are relevant to dynamics driven by a large class of network based matrices, including the Laplacian and adjacency matrices. Redundancy of the control product is addressed by exploiting the graph automorphisms of the network, similar to [5] and [10]. The utility of this approach is demonstrated through a distributed scheme for output feedback stabilization and a social network application.

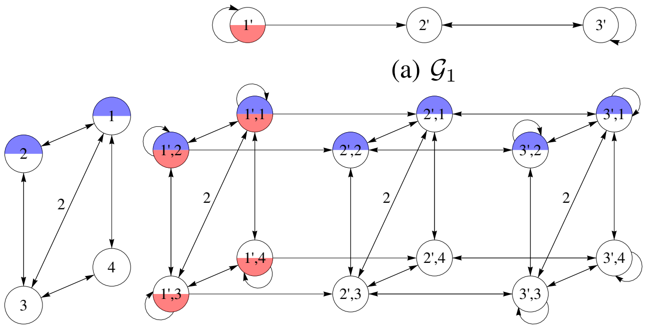
The organization of the paper is as follows. We begin by introducing relevant background material pertaining to graphs, Cartesian products, Kronecker sums and graph automorphisms. We describe the network based state dynamics over a large class of matrix representations of graphs. We then provide a control scheme that forms a controllable state matrix for a composite graph from the controllable state matrices of the factor graphs. Symmetries in the graph provide a sufficient condition for the control scheme to produce the minimal number of controllable input nodes. A second control scheme extends the control of a single factor graph to the composite graph by exploiting the layering structure of the Cartesian product. A layered output feedback controller is presented pertaining to the layered control scheme. Finally, we examine the problem of estimating the opinion dynamics of a social influence network utilizing the observability properties of the Cartesian product.

II. Notation and Background

We provide the notation and a brief background on the constructs and models that will be used in this paper.

1) General Notation: For a column vector $v \in \mathbb{R}^p$, both v_i and $[v]_i$ denote its ith element. For matrix $M \in \mathbb{R}^{p\times q}$, $[M]_{ij}$ denotes the element in its ith row and jth column. The $n \times n$ identity matrix is denoted I_n and e_i is the column vector with all zero entries except $[e_i]_i = 1$.

2) Graphs: A weighted digraph $\mathcal{G} = (V, E, W)$ is characterized by a node set V with cardinality n, an edge set E comprised of ordered pairs of nodes with cardinality m, and a weight set W with cardinality m, where an edge exists from node i to j if $(i, j) \in E$ with edge weight $w_{ji} \in W$. The adjacency matrix of $\mathcal{G}$, denoted $\hat{E}(\mathcal{G})$, is an $n \times n$ matrix with $[\hat{E}(\mathcal{G})]_{ij} = w_{ij}$ when $(j, i) \in E$ and $[\hat{E}(\mathcal{G})]_{ij} = 0$ otherwise. The self-loop matrix $\Delta_s(\mathcal{G}) \in \mathbb{R}^{n\times n}$ is a diagonal matrix with w_{ii} at position (i, i). The in-degree



(a) $\mathcal{G}_1$

(b) $\mathcal{G}_2$ (c) $\mathcal{G} = \mathcal{G}_1 \square \mathcal{G}_2$

Fig. 1. Factor graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ and composite graph $\mathcal{G}_1 \square \mathcal{G}_2$. Edge weights of all graphs are 1 unless otherwise marked. The shading on the nodes pertains to Example IV.2 and V.2.

matrix $\Delta_{in}(\mathcal{G}) \in \mathbb{R}^{n\times n}$ is a diagonal matrix with the number of edges incident to node i, its in-degree, at position (i,i). The out-degree matrix $\Delta_{out}(\mathcal{G})$ is similarly defined.

3) Cartesian Products: There are many effective methods via which large-scale networks (graphs) can be synthesized from a set of smaller graphs [15]. The Cartesian product is one such method and is defined for a pair of *factor* graphs $\mathcal{G}_1 = (V_1, E_1, W_1)$ and $\mathcal{G}_2 = (V_2, E_2, W_2)$ and denoted by $\mathcal{G} = \mathcal{G}_1 \square \mathcal{G}_2$. The product graph $\mathcal{G}$ has the node set $V_1 \times V_2$ and there is an edge from node (i,p) to (j,q) in $V_1 \times V_2$ if and only if either $i = j$ and (p,q) is an edge of E_2, or $p = q$ and (i,j) is an edge of E_1. The corresponding weight, if an edge exists, is $w_{((j,q),(i,p))} = w_{ji}^{\delta_{pq}} + w_{qp}^{\delta_{ij}}$, where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. An example of a Cartesian product of two factor graphs is displayed in Figure 1.

A graph is called *prime* if it cannot be decomposed into the product of non-trivial graphs, otherwise a graph is referred to as composite. All graphs have a *prime factor decomposition* of the form $\mathcal{G}_1^{k_1} \square \cdots \square \mathcal{G}_m^{k_m}$, where $\mathcal{G}_i$ is prime for all i and $\mathcal{G}_i^{k_i}$ denotes k_i Cartesian products of $\mathcal{G}_i$. Cartesian products can be decomposed uniquely into primes in polynomial-time [15].

4) Kronecker Sums: Of particular interest to this paper is the Kronecker sum defined on square matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times m}$ as $A \oplus B := A \otimes I_m + I_n \otimes B$, where $\otimes$ is the Kronecker product. A property of the Kronecker sum is that given the left eigenvalue-eigenvector pairs of A and B as (λ_i, u_i) for $i = 1, \ldots, n$ and (μ_j, v_j) for $j = 1, \ldots, m$, respectively, then $(\lambda_i + \mu_j, u_i \otimes v_j)$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ are left eigenvalue-eigenvector pairs of $A \oplus B$. If A and B are diagonalizable matrices their eigenvectors span $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Consequently, $u_i \otimes v_j$, for all i and j form a spanning set of eigenvectors of $A \oplus B$ as $\textbf{rank}(U \otimes V) = \textbf{rank}(U)\textbf{rank}(V)$, for $U = [u_1, \ldots, u_n]$ and $V = [v_1, \ldots, v_m]$. Thus, if $\lambda_i + \mu_j$ is simple, A and B are diagonalizable and $u_i \otimes v_j$ is a left eigenvector if and only if (λ_i, u_i) and (μ_i, v_j) are left eigenvalue-eigenvectors of A and B, respectively.

5) Graph Automorphisms: Formally, a graph automorphism is a permutation σ of the node set such that $\mathcal{G}$ contains an edge (i,j) with weight w_{ji} if and only if it also contains an edge $(\sigma(i), \sigma(j))$ with weight $w_{\sigma(j)\sigma(i)}$. The set of automorphisms, which forms a group, is denoted as $\text{Aut}(\mathcal{G})$. Every graph automorphism can be represented uniquely as a permutation matrix J which commutes with the adjacency matrix, i.e., $J\hat{E}(\mathcal{G}) = \hat{E}(\mathcal{G})J$. An automorphism σ *fixes* node i if $\sigma(i) = i$.

We now proceed to introduce the system dynamics and its relationship to the underlying graph.

III. Problem Setup

There are a number of ways to construct a matrix $A(\mathcal{G}) \in \mathbb{R}^{n\times n}$ from the edges and nodes of an n node graph $\mathcal{G}$. Some examples we have already touched upon are the adjacency, self-loop, in-degree and out-degree matrices.

One of the properties common to all of the aforementioned matrix representations is that they preserve the symmetries in the graph. By this we mean that for a representations $A(\cdot)$ if there exists a permutation matrix J corresponding to some graph automorphism of $\mathcal{G}$ then $A(\mathcal{G})J = JA(\mathcal{G})$. We refer to this matrix representations feature as *symmetry preserving*.

In this paper, we will be considering the specific family of symmetry preserving matrix representations which also exhibit the added property that the representation is invariant under the Cartesian product. Formally, these representations satisfy

$$A(\mathcal{G}_1 \square \mathcal{G}_2) = A(\mathcal{G}_1) \oplus A(\mathcal{G}_2),$$

for all graphs $\mathcal{G}_1$ and $\mathcal{G}_2$. In general, any matrix representation of the form[1]

$$[A(\mathcal{G})]_{ij} = \begin{cases} rw_{ii} + \sum_{j\neq i} f(w_{ij}, w_{ji}) & \text{for } i = j \\ g(w_{ij}, w_{ji}) & \text{otherwise,} \end{cases}$$

where $r \in \mathbb{R}$ and $f(\cdot)$ and $g(\cdot)$ are real-valued functions such that $f(0,0) = g(0,0) = 0$, satisfies these properties.We denote this family of matrix representation as $\boldsymbol{A}_{\oplus}$.

All matrix representations so far are members of $\boldsymbol{A}_{\oplus}$. Another well known member of $\boldsymbol{A}_{\oplus}$ is the in-degree graph Laplacian (or Laplacian) matrix $\mathcal{L}(\mathcal{G})$ defined as $[\mathcal{L}(\mathcal{G})]_{ij} = -[\hat{E}(\mathcal{G})]_{ij}$ for $i \neq j$ and $[\mathcal{L}(\mathcal{G})]_{ii} = [\Delta_{in}(\mathcal{G})]_{ii}$. The Laplacian matrix features in the popular consensus dynamics and, with the exception of self-loops, uniquely defines the graph $\mathcal{G}$. Other noteworthy members are the out-degree graph Laplacian $\mathcal{L}_{out}(\mathcal{G})$ defined as $[\mathcal{L}_{out}(\mathcal{G})]_{ij} = -[\hat{E}(\mathcal{G})]_{ij}$ for $i \neq j$ and $[\mathcal{L}_{out}(\mathcal{G})]_{ii} = [\Delta_{out}(\mathcal{G})]_{ii}$ and used in advection dynamics [16] and the M-matrix representation $M(\mathcal{G})$ where $[M(\mathcal{G})]_{ij} = -[\hat{E}(\mathcal{G})]_{ij}$ for $i \neq j$ and $[M(\mathcal{G})]_{ii} = [\hat{E}(\mathcal{G})]_{ii}$ investigated in [17]. The class $\boldsymbol{A}_{\oplus}$ of representations is by no means a small one and other members will be featured in examples throughout the paper. It is easy to show that $\boldsymbol{A}_{\oplus}$ is closed under addition providing a simple mechanism to generate new members.

In this paper, we explore controllability and observability of systems of the form

$$\dot{x}(t) = A(\mathcal{G})x(t) + Bu(t), \quad y(t) = Cx(t), \qquad (1)$$

where $A(\cdot) \in \boldsymbol{A}_{\oplus}$. For brevity, we refer to these dynamics by defining the matrix triplet $(A(\mathcal{G}), B, C)$, or if only the

[1] Here we assume that if there is no edge $(i,j) \in E$ then $w_{ji} = 0$.

inputs (outputs) are of interest by the matrix pair $(A(\mathcal{G}), B)$ $((A(\mathcal{G}), C))$.

It should be mentioned that, due to the linear system duality between controllability and observability, the pair $(A(\mathcal{G}), B)$ is controllable if and only if the pair $(A(\mathcal{G}), B^T)$ is observable. Hence, the results throughout this paper will be presented in terms of controllability but are equally applicable to the network observability problem.

A helpful tool to establish controllability is the well known Popov-Belevitch-Hautus (PBH) test [18]. The test states that (A, B) is uncontrollable if and only if there exists a left eigenvalue-eigenvector pair (λ, v) of A, such that $v^T B = 0$.

It is often of interest where the inputs and outputs of system (1) are in terms of the nodes of the graph $\mathcal{G}$. If the set of input nodes in the n node graph is $S = \{i_1, i_2, \ldots, i_p\}$ for $i_1 < i_2 < \cdots < i_p$, the corresponding input matrix is $B = \begin{bmatrix} e_{i_1}, e_{i_2}, \ldots, e_{i_p} \end{bmatrix} \in \mathbb{R}^{n \times p}$. We uniquely denote the input matrices of this form as $B_n(S)$. Similarly, the output matrices are defined as $C_n(S) := B_n(S)^T$. If it is clear from the context, we remove the subscript n for brevity.

IV. Control Product

The following theorem introduces our first method of extending control of the factors of the composite graph $\mathcal{G}$ to $\mathcal{G}$ itself for the case where $A(\mathcal{G})$ has simple eigenvalues. Specifically, we examine the controllability of the pair $(A(\mathcal{G}), B)$, first discussed in [5], where $\mathcal{G} = \mathcal{G}_1 \Box \mathcal{G}_2$ and $B = B_1 \otimes B_2$.

Theorem IV.1. *Consider $A(\cdot) \in \boldsymbol{A}_\oplus$ and graphs $\mathcal{G}_1$ and $\mathcal{G}_2$. Assume $A(\mathcal{G}) = A(\mathcal{G}_1 \Box \mathcal{G}_2)$ has simple eigenvalues. Then, the pairs $(A(\mathcal{G}_1), B_1)$ and $(A(\mathcal{G}_2), B_2)$ are controllable if and only if $(A(\mathcal{G}), B)$ is controllable, where $B = B_1 \otimes B_2$.*

Proof: Assuming the eigenvalues of $A(\mathcal{G})$ are simple, w is a left eigenvector of $A(\mathcal{G})$ if and only if $w = u \otimes v$ for some left eigenvector u and v of $A(\mathcal{G}_1)$ and $A(\mathcal{G}_2)$, respectively (see §II-4). Hence, $w^T B = (u \otimes v)^T (B_1 \otimes B_2) = (u^T \otimes v^T)(B_1 \otimes B_2) = u^T B_1 \otimes v^T B_2$. Now, $w^T B = 0$ if and only if $u^T B_1 = 0$ or $v^T B_2 = 0$ (or both). Thus, by the PBH test, $(A(\mathcal{G}), B)$ is controllable if and only if $(A(\mathcal{G}_1), B_1)$ and $(A(\mathcal{G}_2), B_2)$ are controllable. ∎

Consider the case where $B_1 := B(S_1)$ and $B_2 := B(S_2)$, then for $S = S_1 \times S_2 := \{(i, j) | i \in S_1 \text{ and } j \in S_2\}$ we have $B(S) = B_1 \otimes B_2$. This motivates the name *control product* for the scheme.

The following example demonstrates Theorem IV.1 and the form of set S.

Example IV.2. *Consider graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ in Figure 1. Let the eigenvalues of $\mathcal{L}(\mathcal{G}_1)$ be $\lambda_1, \lambda_2, \lambda_3$ and similarly the eigenvalues for $\mathcal{L}(\mathcal{G}_2)$ be $\mu_1, \mu_2, \mu_3, \mu_4$. As $\lambda_i + \mu_j$ for $i = 1, \ldots, 3$ and $j = 1, \ldots 4$ are the eigenvalues of $\mathcal{L}(\mathcal{G})$, it is efficient to check that the eigenvalues are distinct. For $S_1 = \{1'\}$ and $S_2 = \{1, 2\}$, the pairs $(-\mathcal{L}(\mathcal{G}_1), B_3(S_1))$ and $(-\mathcal{L}(\mathcal{G}_2), B_4(S_2))$ are controllable. The nodes corresponding to sets S_1 and S_2 are half shaded in Figure 1. Now, $B_3(S_1) \otimes B_4(S_2) = B_{12}(S)$ where $S = \{(1', 1), (1', 2)\}$, denoted by full shaded nodes in Figure 1. Therefore from Theorem IV.1, $(-\mathcal{L}(\mathcal{G}), B_{12}(S))$ is controllable.*

We now provide an example to illustrate the requirement that the composite graph in Theorem IV.1 has simple eigenvalues.

Example IV.3. *Denote the path graph of length 2 as $\mathcal{P}_2$. It is a well known property that $\mathcal{L}(\mathcal{P}_2)$ has simple eigenvalues 0 and 2 and for $S = \{1\}$, $(-\mathcal{L}(\mathcal{P}_2), B_2(S))$ is controllable. Further, $\mathcal{P}_2 \Box \mathcal{P}_2 = \mathcal{C}_4$ a length four cycle graph, with non-distinct eigenvalues 0, 2, 2 and 4, and $B_2(S) \otimes B_2(S) = B_4(S)$. But all cycle graphs are uncontrollable from one node (see [10], [11]) so $(-\mathcal{L}(\mathcal{C}_4), B_4(S))$ is uncontrollable.*

As an aside, the distinctness of the eigenvalues of $A(\mathcal{G})$ has implication on the structure of $\mathcal{G}$, for example on its prime factor decomposition described in the following proposition, a feature which will be applied later.

Proposition IV.4. *For $A(\cdot) \in \boldsymbol{A}_\oplus$, if the eigenvalues of $A(\mathcal{G})$ are simple, then the prime factor decomposition of $\mathcal{G}$ contains no powers of prime graphs.*

Proof: Let the prime factor decomposition of the n node graph $\mathcal{G}$ be $\mathcal{G} = \mathcal{G}_1^{k_1} \Box \cdots \Box \mathcal{G}_m^{k_m}$. If $\mathcal{G}$ is a power of a prime graph then there exists a $k_i \geq 2$ corresponding to the m node prime factor $\mathcal{G}_i$. Hence, $\mathcal{G} = \mathcal{G}_i^2 \Box \mathcal{G}_b$ where $\mathcal{G}_b = \mathcal{G}_1^{k_1} \Box \cdots \Box \mathcal{G}_i^{k_i - 2} \Box \cdots \Box \mathcal{G}_m^{k_m}$. Therefore $A(\mathcal{G})$ has pairs of eigenvalues with the value $\lambda_i + \lambda_j + \mu_k$ for $i, j = 1, \ldots, m$, $i \neq j$ and $k = 1, \ldots, \frac{n}{m}$, where λ_i and λ_j are eigenvalues of $A(\mathcal{G}_i)$, and μ_k is an eigenvalue of $A(\mathcal{G}_b)$. This follows from the eigenvalue properties of the Cartesian sum. ∎

A. Breaking Symmetry

The automorphisms of a graph describe its symmetries and have been previously shown to play an important role in the controllability of $(-\mathcal{L}(\mathcal{G}), B(S))$ for an undirected, unweighted graph $\mathcal{G}$ [5]. The following is a generalization of these results found in [5] and so is quoted here without proof.

Proposition IV.5. *A graph $\mathcal{G}$ is uncontrollable from any pair $(A(\mathcal{G}), B(S))$ where $A(\cdot)$ is symmetry preserving and $A(\mathcal{G})$ has spanning eigenvectors, if there exists an automorphism of $\mathcal{G}$ which fixes all inputs in the set S.*

We note that Proposition IV.5 is not sufficient for controllability as discussed in [5]. Further, this proposition is applicable to Theorem IV.1, as if a matrix has simple eigenvalues then its eigenvectors are spanning.

Proposition IV.5 highlights the requirement of selecting a set of inputs that break the symmetry structure of $\mathcal{G}$. Determining sets are a useful construct to describe this process.

Definition IV.6. *A subset S of the vertices of a graph $\mathcal{G}$ is called a determining set if whenever $g, h \in Aut(\mathcal{G})$ so that $g(s) = h(s)$ for all $s \in S$, then $g = h$. The determining number of a graph $\mathcal{G}$, denoted $Det(\mathcal{G})$, is the smallest integer r so that $\mathcal{G}$ has a determining set of size r.*

Another term for a determining set S is the *fixing set* due to the fact that no non-trivial automorphism of $\mathcal{G}$ fixes all members in S. Formally, a set $S \subseteq V(\mathcal{G})$ is a determining set if and only if the stabilizing set $Stab(S)$ of S, defined as $Stab(S) := \{g \in \text{Aut}(\mathcal{G}) | \sigma(v) = v, \forall \sigma \in S\}$, only contains the trivial automorphism. Directly from Proposition IV.5 and the definition of determining sets we have the following corollary.

Corollary IV.7. *A necessary condition for controllability of the pair $(A(\mathcal{G}), B(S))$ is that S is a determining set. Hence, $|S| \geq Det(\mathcal{G})$.*

The automorphism group of a composite graph is intimately linked to the automorphisms of its prime factors. This link translates through to the determining set of the composite graph summarized in the following:

Theorem IV.8. *[19] Let $\mathcal{G} = \mathcal{G}_1^{k_1} \Box \cdots \Box \mathcal{G}_m^{k_m}$ be the prime factor decomposition for a connected graph $\mathcal{G}$. Then $Det(\mathcal{G}) = \max\{Det(\mathcal{G}_i^{k_i})\}$.*

We now have the required ground work to state a consequence of the graph automorphism structure of the graph pertaining to Theorem IV.1.

Theorem IV.9. *Under the assumptions of Theorem IV.1, consider the controllable pairs $(A(\mathcal{G}_1), B(S_1))$ and $(A(\mathcal{G}_2), B(S_2))$ where $|S_1| = Det(\mathcal{G}_1)$ and $|S_2| = 1$. Then $S = S_1 \otimes S_2$ is the smallest set such that $(A(\mathcal{G}_1 \Box \mathcal{G}_2), B(S))$ is controllable.*

Proof: As all eigenvalues of $A(\mathcal{G}_1 \Box \mathcal{G}_2)$ are simple, from Proposition IV.4 the prime factors of $\mathcal{G}$, and subsequently $\mathcal{G}_1$ and $\mathcal{G}_2$, are relatively prime. Thus from Theorem IV.8, $Det(\mathcal{G}) = \max(Det(\mathcal{G}_1), Det(\mathcal{G}_2))$. Further, as $(A(\mathcal{G}_2), B(S_2))$ is controllable then $1 = |S_2| \geq Det(\mathcal{G}_2) \geq 1$, so $Det(\mathcal{G}) = Det(\mathcal{G}_1)$. Now $B(S_1) \otimes B(S_2) = B(S)$ for some $S \subseteq V(\mathcal{G}_1 \Box \mathcal{G}_2)$ and $|S| = |S_1||S_2| = |S_1|$. As $|S| = Det(\mathcal{G})$, the pair $(A(\mathcal{G}_1 \Box \mathcal{G}_2), B(S))$ is controllable with the smallest number of inputs. ■

We now revisit Example IV.2 with Theorem IV.9 in mind.

Example IV.10. *Further examination of Example IV.2, the $Aut(\mathcal{G}_2) = \{id, \sigma, \tau, \sigma\tau\}$ where id is the identity permutation, $\sigma(1, 2, 3, 4) = (1, 4, 3, 2)$ and $\tau(1, 2, 3, 4) = (3, 2, 1, 4)$. Hence, $Det(\mathcal{G}_2) = 2 = |S_2|$ and $|S_1| = 1$. Applying Theorem IV.9, S is the smallest controllable input set.*

V. Layered Control

The following theorem details our second method for extending control of the factors to the composite graph. This control scheme involves repeating the form of control matrix B_1 to every $\mathcal{G}_1$ layer of $\mathcal{G}$, motivating the name *layered control*. As the Kronecker product exhibits permutation equivalency, these results are equivalent to extending the control matrix B_2 to every $\mathcal{G}_2$ layer of $\mathcal{G}$.

Theorem V.1. *Consider $A(\cdot) \in \boldsymbol{A}_{\oplus}$ and graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ in Figure 1 with n_1 and n_2 nodes, respectively, where the matrices $A(\mathcal{G}_1)$ and $A(\mathcal{G}_2)$ are diagonalizable. The pair $(A(\mathcal{G}_1), B_1)$ is controllable if and only if $(A(\mathcal{G}), B)$ is controllable, where $\mathcal{G} = \mathcal{G}_1 \Box \mathcal{G}_2$ and $B = B_1 \otimes I_{n_2}$.*

Proof: (*Only If*) Assume $(A(\mathcal{G}_1), B_1)$ is controllable; then by the PBH test there exists some left eigenvector u such that $u^T B_1 = 0$. From §II-4, $u \otimes v$ is a left eigenvector of $(A(\mathcal{G}), B)$, where v is some left eigenvector of $(A(\mathcal{G}_2), B_2)$, and $(u \otimes v)^T B = (u^T \otimes v^T)(B_1 \otimes I_{n_2}) = u^T B_1 \otimes v^T = 0$. Therefore, $(A(\mathcal{G}), B)$ is uncontrollable.

(*If*) Assume $(A(\mathcal{G}), B)$ is uncontrollable then there exists some left eigenvector w such that $w^T B = 0$. From §II-4, w is a linear combination of the set of eigenvectors of the form $u_i \otimes v_j$, where u_i and v_j are left eigenvectors of $(A(\mathcal{G}_1), B_1)$ and $(A(\mathcal{G}_2), B_2)$, respectively. Thus, $w = \sum \alpha_{ij}(u_i \otimes v_j)$, where the scalars $\alpha_{ij} \in \mathbb{R}$ are nonzero, and

$$\begin{aligned} 0 &= w^T B = \sum \alpha_{ij}(u_i \otimes v_j)^T (B_1 \otimes I_{n_2}) \\ &= \sum \alpha_{ij}(u_i^T \otimes v_j^T)(B_1 \otimes I_{n_2}) \\ &= \sum \alpha_{ij} u_i^T B_1 \otimes v_j^T, \end{aligned}$$

or equivalently, $\sum \left[\alpha_{ij} u_i^T B_1\right]_k v_j^T = 0$, for $k = 1, \ldots, n_1$. This occurs only if $\left[\alpha_{ij} u_i^T B_1\right]_k = 0$ for all k and i corresponding to some α_{ij}, i.e., $u_i^T B_1 = 0$, since the set of eigenvectors v_j are linearly independent. Therefore, $(A(\mathcal{G}_2), B_2)$ is uncontrollable. ■

The assumption of diagonalizability is, for example, satisfied for invertible, real symmetric and irreducible matrices, as well as matrices with simple eigenvalues. Further, if $\mathcal{G}$ is undirected or strongly connected[2] then $A(\mathcal{G})$ is a real symmetric or irreducible matrix, respectively - satisfying the assumption. This assumption is removed in the extended journal version of this paper [1].

An illustrative example of Theorem V.1 follows.

Example V.2. *Consider graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ in Figure 1. For $S_1 = \{1'\}$ and $S_2 = \{1, 2\}$, the pairs $(-\mathcal{L}(\mathcal{G}_1), B_3(S_1))$ and $(-\mathcal{L}(\mathcal{G}_2), B_4(S_2))$ are controllable, and $\mathcal{L}(\mathcal{G}_1)$ and $\mathcal{L}(\mathcal{G}_2)$ are diagonalizable. The nodes corresponding to sets S_1 and S_2 are half shaded in Figure 1. Now, $B_3(S_1) \otimes I = B_{12}(S_a)$ where $S_a = \{(1', 1), (1', 2), (1', 3), (1', 4)\}$, denoted by the lower half shaded nodes in Figure 1. Similarly, $I \otimes B_4(S_2) = B_{12}(S_b)$ and $S_b = \{(1', 1), (1', 2), (2', 1), (2', 2), (3', 1), (3', 2)\}$, denoted by the upper half shaded nodes in Figure 1. Therefore from Theorem V.1, pairs $(-\mathcal{L}(\mathcal{G}), B_{12}(S_a))$ and $(-\mathcal{L}(\mathcal{G}), B_{12}(S_b))$ are controllable.*

Theorem V.1 provides a useful tool of combining families of graphs with known controllability, such as the Laplacian of the path and cycle graphs, with graphs where controllability is hard to establish, such as random graphs, or graphs that are difficult to control, such as the complete graph.

Further, Theorem V.1 can be combined with Theorem IV.1 to produce controllable graphs. A composite graph $\mathcal{G}$ can be decomposed into $\mathcal{G}_a \Box \mathcal{G}_b$ where $\mathcal{G}_a$ is the largest factor graph

[2]A graph is undirected if $(i, j) \in E$ implies $(j, i) \in E$ and $w_{ij} = w_{ji}$ and strongly connected if between every pair of distinct nodes there exists a directed path.

of $\mathcal{G}$ such that $A(\mathcal{G}_a)$, where $A(\cdot) \in \mathbf{A}_\oplus$, has simple eigenvalues and $\mathcal{G}_b$ has order n_b. Hence, $\mathcal{G}_a$ can be decomposed into its primes factors $\mathcal{G}_1 \square \ldots \square \mathcal{G}_k$. Assuming controllable matrix pairs $(A(\mathcal{G}_i), B_i)$ for $i = 1, \ldots, k$ can be found, then by Theorem IV.1, $(A(\mathcal{G}_a), B_1 \otimes \cdots \otimes B_k)$ is controllable. By Theorem V.1, assuming $A(\mathcal{G}_b)$ is diagonalizable, then $(A(\mathcal{G}), B_1 \otimes \cdots \otimes B_k \otimes I_{n_b})$ is controllable. This technique is used in the following example to establish controllability of the grid $\mathcal{P}_2 \square \mathcal{P}_4 \square \mathcal{P}_5$.

Example V.3. *Denote the path graphs of length two, four, and five path graph as $\mathcal{P}_2, \mathcal{P}_4$, and $\mathcal{P}_5$, respectively. Since all path graphs are controllable from either end node, designating one of the ends as the first node, the pairs $(-\mathcal{L}(\mathcal{P}_4), B_4(S))$ and $(-\mathcal{L}(\mathcal{P}_5), B_5(S))$ are controllable for $S = \{1\}$. Noting that $\mathcal{L}(\mathcal{P}_4 \square \mathcal{P}_5)$ has distinct eigenvalues using the technique as Example IV.2, from Theorem IV.1, the pair $(-\mathcal{L}(\mathcal{P}_4 \square \mathcal{P}_5), B_{20}(S))$ is controllable. Further applying Theorem V.1, and noting all graphs involved are undirected which satisfies the diagonalizable assumption, then $(-\mathcal{L}(\mathcal{P}_2 \square \mathcal{P}_4 \square \mathcal{P}_5), B_{40}(S'))$ for $S' = \{(1,1),(2,1)\}$ is controllable as $I_2 \otimes B_{20}(S) = B_{40}(S')$. Here we have a 40 node grid controllable from 2 nodes.*

Interestingly, $\mathcal{L}(\mathcal{P}_2 \square \mathcal{P}_4 \square \mathcal{P}_5)$ has repeated eigenvalues and so, from the PBH test, at least two input nodes are required to form a controllable set. Hence, the set S' found in Example V.3 is the smallest input set.

A. Layered Output Feedback

An attraction of composite networks is that they exhibit repeated layers of the factors. Theorem V.1 takes advantage of this by extending the controllable inputs in one factor layer to many. The same can be done to the observable outputs. The next proposition shows that the control signal can similarly be designed for a factor and extended to the composite network with the effect of generating distributed output feedback stabilization.

Proposition V.4. *Consider $A(\cdot) \in \boldsymbol{A}_\oplus$ and the n_1 node and n_2 node graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ in Figure 1, where the matrix $A(\mathcal{G}_2)$ is semistable. If the dynamics $(A(\mathcal{G}_1), B_1, C_1)$ is stabilizable with output feedback $u_a = K y_a$ for inputs u_a and outputs y_b then the dynamics $(A(\mathcal{G}_1 \square \mathcal{G}_1), B_1 \otimes I_{n_2}, C_1 \otimes I_{n_2})$ is stabilizable with the output feedback $u = (K \otimes I_{n_2}) y$ for inputs u and outputs y. Further, the control law can be realized with local layer feedback across the layers of $\mathcal{G}_1$.*

Proof: For convenience, we present the equivalent result in terms of the layers of $\mathcal{G}_2$. As K is a stabilizing feedback gain for the system described by the matrices $(A(\mathcal{G}_2), B_2, C_2)$ then $A(\mathcal{G}_2) + B_2 K C_2$ is stable. Consider the dynamics of the system $(A(\mathcal{G}_1 \square \mathcal{G}_2), I_{n_1} \otimes B_2, I_{n_1} \otimes C_2)$ with output feedback $u = (I_{n_1} \otimes K) y$. Then,

$$\begin{aligned}
&\dot{x}(t) \\
&= \left(A(\mathcal{G}_1 \square \mathcal{G}_2) + (I_{n_1} \otimes B_2)(I_{n_1} \otimes K)(I_{n_1} \otimes C_2)\right) x(t) \\
&= \left(A(\mathcal{G}_1) \otimes I_{n_2} + I_{n_1} \otimes A(\mathcal{G}_2) + I_{n_1} \otimes B_2 K C_2\right) x(t) \\
&= \left(A(\mathcal{G}_1) \oplus (A(\mathcal{G}_2) + B_2 K C_2)\right) x(t).
\end{aligned}$$

As the Cartesian sum of semistable and stable matrices is stable then $I_{n_1} \otimes K$ is stabilizing, since each eigenvalue of the composite matrix lies in the left half plane. Further as $I_{n_1} \otimes K$ is block diagonal the feedback loop can be broken into the inputs and outputs of each layer of $\mathcal{G}_2$. Specifically distributing the inputs and outputs, we have $u = \left[u_1^T, \ldots, u_{n_1}^T\right]^T$ and $y = \left[y_1^T, \ldots, y_{n_1}^T\right]^T$, where u_i and y_i are the inputs and outputs of the ith layer of $\mathcal{G}_2$, respectively. Hence, the feedback can be written as $u_i = K y_i$ for $i = 1, \ldots, n_1$, i.e., local layer feedback. ■

Proposition V.4 describes a setup where we have a stabilizing distributed feedback on each factor layer requiring only local feedback on the sensors and actuators placed on that layer. The following is an example illustrating this layered output feedback stabilization.

Example V.5. *Define the matrix representation*

$$[A(\mathcal{G})]_{ij} = \begin{cases} w_{ij} & \text{for } i \neq j \\ \frac{1}{2} w_{ii} - \sum_{i \neq j} w_{ij} & \text{otherwise.} \end{cases}$$

An equivalent form is $A(\mathcal{G}) := -L(\mathcal{G}) + \frac{1}{2}\Delta_s$, and so $A(\cdot) \in \boldsymbol{A}_\oplus$. For the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ described in Figure 1, $A(\mathcal{G}_1)$ is unstable and $A(\mathcal{G}_2)$ is semistable. Consider the dynamics of the system described by the matrices $(A(\mathcal{G}_1), B(S_1), C(S_2))$ where $S_1 = \{1'\}$ and $S_2 = \{2'\}$ then the output feedback $u = ky$ is stabilizing for $k < -\frac{1}{2}$. From Proposition V.4, the output feedback $k \otimes I_4$ is stabilizing for the composite system, which is realized by the distributed feedback $u_{(1',i)} = k y_{(2',i)}$ for $i = 1, \ldots, 4$, where $u_{(1',i)}$ is the input applied to node $(1', i)$ and $y_{(2',i)}$ is the output measured from node $(2', i)$.

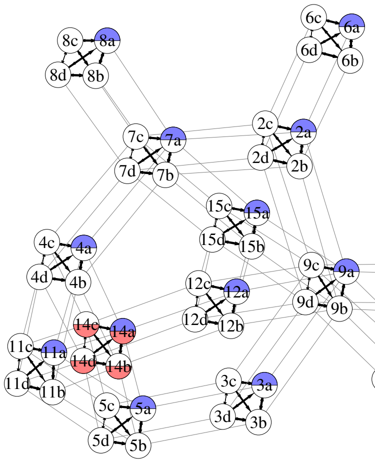
VI. Filtering on Social Product Networks

A Cartesian network structure is not uncommon in social networks due to the layered structure of the society. For example, consider an interacting network of nuclear families in a neighborhood. Interactions among families often involve like-gendered parents interacting with like-gendered parents and similarly like-aged children with like-aged children. This network can be realized through a Cartesian product $\mathcal{G}_1 \square \mathcal{G}_2$ of the inter-family interaction graph $\mathcal{G}_1$ and the family members interaction graph $\mathcal{G}_2$.

It is often unrealistic or expensive to make a census of the full population of a social group. An alternative is to sample the network, and subsequently estimate its state dynamics through an *opinion dynamics filter*. However, a requirement for designing such an estimator is observability.

For our example, the underlying model has been adopted from [20], and is the discrete form of the continuous dynamics $(A(\mathcal{G}), C(S))$, where $A(\cdot) = -\mathcal{L}(\cdot) \in \mathbf{A}_\oplus$, $x(t)$ is the n agents' opinions and $\mathcal{G} = \mathcal{G}_1 \square \mathcal{G}_2$ is the underlying influence network and S is the set of sampled agents.

We consider the inter-family interaction graph $\mathcal{G}_1$ based on the famous Florentine family graph [21], with each node representing one of the fifteen families. The layers of $\mathcal{G}_1$ are denoted in Figure 2 by the grey undirected, unweighted edges. The family member interaction graph $\mathcal{G}_2$ with nodes a,



$$\hat{E}(\mathcal{G}_2) = \begin{bmatrix} 0 & 0.4 & 0.1 & 0.1 \\ 0.3 & 0 & 0.2 & 0.2 \\ 0.4 & 0.3 & 0 & 0.1 \\ 0.35 & 0.45 & 0.1 & 0 \end{bmatrix}$$

Fig. 2. Composite graph $\mathcal{G}_1 \square \mathcal{G}_2$. The layers of $\mathcal{G}_1$ are displayed with grey edges and have edge weight 1. The layers of $\mathcal{G}_2$ are black with weights divulged through the definition of $\hat{E}(\mathcal{G}_2)$. The shading on the nodes pertains to the control inputs, S_F (upper half shaded nodes), S_{14} (lower half shaded nodes) and S_{F14} (fully shaded nodes), described in Example VI.

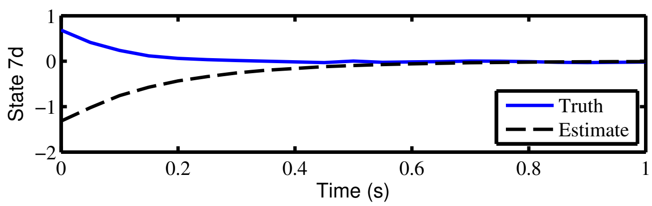


Fig. 3. The true and estimated state of a sample member of the society, namely the youngest child of family 7 (node 7d), over time for the discrete Kalman filter pertaining to Example VI.

b, c and d, correspond to the interaction network in a nuclear family amongst a father, mother, older child and younger child, respectively, with adjacency matrix $\hat{E}(\mathcal{G}_2)$ appearing in Figure 2. The composite graph $\mathcal{G}_1 \square \mathcal{G}_2$ corresponds to the resulting 60 members' interaction graph, and is depicted in Figure 2.

Assuming that all the members of one demographic in the social network can be observed, e.g., all the mothers, then Theorem V.1, applied to $\mathcal{G}_2$, provides the necessary observability condition for the filter design.

For example, the dynamics are observable, under one demographic, for $C_{60}(S) = I \otimes C_4(S_2)$, where $S_2 = \{a\}$ and $S = S_F := \{1a, 2a, \ldots, 15a\}$, the set of fathers of the society. This information would be attractive to an advertiser, as the opinions of all 60 members can be divulged by surveying the fathers. If instead an advertiser was interested in a good family to survey, then Theorem V.1 can be applied to $\mathcal{G}_1$, leading to an observability matrix $C_{60}(S) = C_{15}(S_1) \otimes I$, where $S_1 = \{14\}$ and $S = S_{14} := \{14a, 14b, 14c, 14d\}$, i.e., every member of family 14 is observed directly.

Alternatively, as $A(\mathcal{G}_1 \square \mathcal{G}_2)$ has simple eigenvalues, using Theorem IV.1, the dynamics with the observation matrix $C_{60}(S) = C_{15}(S_1) \otimes C_4(S_2)$ is observable, where $S = S_1 \times S_2 = S_{F14} := \{14a\}$. Therefore, in this scenario, surveying the father of family 14 would provide the opinion of all members.

A discrete Kalman filter was applied to the described social dynamics, with all fathers in the social network observed, i.e., $S = S_F$. A sample opinion state estimate over time is provided in Figure 3, supporting the observability of the pair $(A(\mathcal{G}_1 \square \mathcal{G}_2), C_{60}(S))$.

VII. Conclusion

This paper presents an analysis of the controllability and observability of dynamics over composite networks formed by the graph Cartesian product of its factors. Using the tools of graph theory, group theory, and Kronecker algebra, we explored the composition of the controllable input sets of the factor networks to form a controllable control set of the composite network. Future work of particular interest involves extending these results to other types of graph products such as the direct product.

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