CARTESIAN PRODUCTS OF Z-MATRIX NETWORKS: FACTORIZATION AND INTERVAL ANALYSIS*

AIRLIE CHAPMAN AND MEHRAN MESBAHI†

Abstract. This paper examines the relationship between the dynamics of large networks and of their smaller factor-networks (factors) obtained through the factorization of the network's graph representation. We specifically examine dynamics of networks which have Z-matrix state matrices. We perform a Cartesian product decomposition on its network structure producing factors which also have Z-matrix dynamics. A factorization lemma is presented that represents the trajectories of the large network in terms of the factors' trajectories. An interval matrix lemma provides families of network dynamics whose trajectories are bounded by the interval bounds' factors' trajectories.

Key words. Z-matrices, Cartesian Product, Interval Matrices, Coordination Algorithms

AMS subject classifications. 93B11, 93A14, 15B48

1. Introduction.

Complex dynamic networks are an integral part of the technological world with examples including the internet, power grids, and communication networks, as well as in the world at large such as biological and chemical systems and social networks. An explosion of research in the area of network systems has eventuated [2, 5, 8]. In parallel, Z-matrices are used to model synchronization in networks [10], population migration [6], Markov processes, and supply and demand in economic systems [1].

Z-matrices also appear in the discretization of differential operators. An example is the discretization of diffusion [10] and advection [3] dynamics which generate the in-degree and out-degree Laplacian matrix respectively, and both of which are Z-matrices. The in-degree Laplacian matrix forms the basis of consensus models. These models are effective for both distributed information-sharing and control of networked, multi-agent systems in settings such as multi-vehicle control, formation control, swarming, and distributed estimation; see, for example, [12, 15].

In this work we examine large networks which are Cartesian products of smaller factor-networks (factors). We present a factorization lemma which represents the larger network trajectories in terms of the factors' trajectories, provided certain initial conditions are met. This result is an extension of the related results on Cartesian products over Laplacian-based simple networks for a constrained set of initial conditions by Nguyen and Mesbahi [11]. We extend the factorization lemma to both non-decomposable networks and arbitrary nonnegative initial conditions, bounding the large network trajectories with the factors' trajectories.

The organization of the paper is as follows. We begin by introducing relevant background material pertaining to graphs, Cartesian products and Kronecker products. We introduce the Z-matrix state dynamics. Interval matrices are used to introduce families of Z-matrices with similar trajectories. The Cartesian product over Z-matrices is introduced as a method to decompose large Z-matrix dynamics to smaller Z-matrix factor dynamics. The paper culminates in the presentation of two factorization lemmas. The first lemma allows perfect characterization of the larger network trajectories in terms of the factors' unforced and forced trajectories. The second

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lemma presents a family of larger network trajectories which are bounded by factors' trajectories.

2. Background.

We provide a brief background on constructs and models that will be used in this paper.

For column vector $v \in \mathbb{R}^p$, v_i or $[v]_i$ denotes the ith element. The column vector $\boldsymbol{e}_i$, has 1 in its ith row and 0, otherwise. For matrix $M \in \mathbb{R}^{p\times q}$, $[M]_{ij}$ denotes the element in its ith row and jth column. A matrix M is nonnegative (positive), denoted $M \geq 0$ ($M > 0$) if all entries of M are nonnegative (positive). Further, $M \geq N$ ($M > N$) is equivalent to $M - N \geq 0$ ($M - N > 0$). The Kronecker product of matrices A and B, is denoted by $A \otimes B$ and the Kronecker sum of square matrices $C \in \mathbb{R}^{n\times n}$ and $D \in \mathbb{R}^{m\times m}$ defined and denoted as $C \oplus D := I_m \otimes C + D \otimes I_n$. The matrix exponential, denoted as e^F for a square matrix F, of a Kronecker sum has the attractive distributive property that $e^{C\oplus D} = e^C \otimes e^D$.

2.1. Graphs.

A weighted graph $\mathcal{G} = (V, E, W)$ is characterized by a node set V with cardinality n, an edge set E comprised of ordered pairs of nodes with cardinality m, and a weight set W with cardinality m. The adjacency matrix is an $n \times n$ matrix with $[\mathcal{A}(\mathcal{G})]_{ij} = w_{ij} \in W$ when $(j, i) \in E$ and $[\mathcal{A}(\mathcal{G})]_{ij} = 0$ otherwise. The self-loop matrix $\Delta_s(\mathcal{G}) \in \mathbb{R}^{n\times n}$ is a diagonal matrix with w_{ii} at position (i, i).

A special family of graphs is the strongly connected graphs where a graph is strongly connected if between every pair of distinct nodes there exists a directed path of edges.

2.2. Cartesian Product.

There is an abundance of effective methods via which large-scale networks (graphs) can be synthesized from a set of smaller graphs [7]. The Cartesian product is one such method and is defined for a pair of *factor* graphs $\mathcal{G}_1 = (V_1, E_1, W_1)$ and $\mathcal{G}_2 = (V_2, E_2, W_2)$ and denoted by $\mathcal{G} = \mathcal{G}_1 \square \mathcal{G}_2$. The product graph $\mathcal{G}$ has the vertex set $V_1 \times V_2$ and there is an edge from vertex (i, p) to (j, q) in $V_1 \times V_2$ if and only if either $i = j$ and (p, q) is an edge of E_2, or $p = q$ and (u, v) is an edge of E_1. The corresponding weight if an edge exists is $w_{((i,p),(j,q))} = w_{ij}^{\delta_{pq}} + w_{pq}^{\delta_{ij}}$ where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. The Cartesian product is commutative and associative, i.e., the products $\mathcal{G}_1 \square \mathcal{G}_2$ and $\mathcal{G}_2 \square \mathcal{G}_1$ are isomorphic; similarly $(\mathcal{G}_1 \square \mathcal{G}_2) \square \mathcal{G}_3$ and $\mathcal{G}_1 \square (\mathcal{G}_2 \square \mathcal{G}_3)$ are isomorphic.

An example of the Cartesian product of two factor graphs is displayed in Figure 2.1a)-c).

A graph is called *prime* if it cannot be decomposed into the product of non-trivial graphs, otherwise a graph is referred to as composite. Sabidussi [14] and Vizing [16] highlighted the fundamental nature of the primes, and noted that connected graphs decompose uniquely into primes, up to reordering. Further, Feigenbaum [4] demonstrated that a graph can be factored into primes in polynomial-time.

Many features of the factors of a composite graph transfer to the composite graph itself. One such example is that if factors $\mathcal{G}_1$ and $\mathcal{G}_2$ are strongly connected, then so too is $\mathcal{G}_1 \square \mathcal{G}_2$. In this paper we show that when the composite graph underlies a dynamic system that many useful features of dynamics can be revealed by examining dynamics systems over the factor graphs.

We now proceed to introduce Z-matrix dynamics and their underlying graph.

3. Z-matrix Dynamics.

We consider a multi-agent network of n coupled nodes with the state of each node i defined as $x_i(t) \in \mathbb{R}$ at time t, and driven by a control

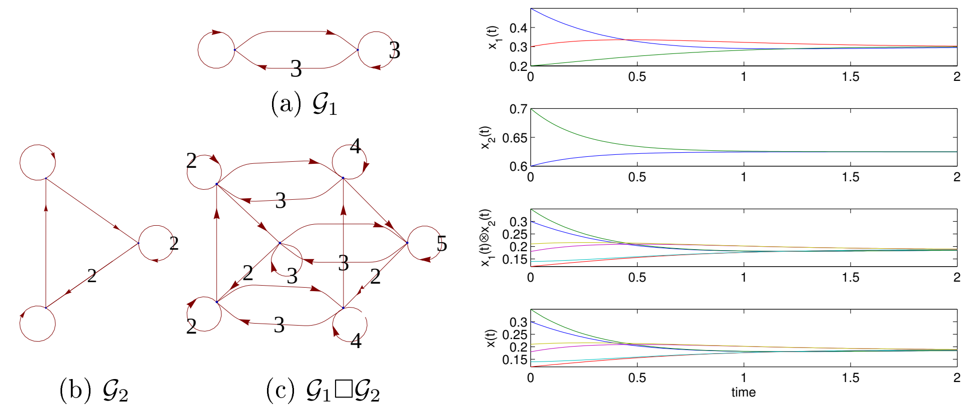

FIGURE 2.1. *Left: Factor graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ and composite graph $\mathcal{G}_1 \square \mathcal{G}_2$. Edge weights of all graphs are 1 unless otherwise marked. Right: Trajectories of the Z-matrix dynamics over $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_1 \square \mathcal{G}_2$ with unforced state dynamics $x_1(t)$, $x_2(t)$ and $x(t)$, respectively, compared to $x_1(t) \otimes x_2(t)$; for details see Lemma 5.2.*

$u(t) \in \mathbb{R}^m$. The system is described by the differential equations

$$\dot{x}_i(t) = -w_{ii} x_i(t) + \sum_{j \in \mathcal{N}(i)} w_{ij} x_j(t) + b_i^T u(t), \; i = 1, \dots, n.$$

The unforced and forced components of the signal $x(t)$ are denoted by $x_u(t)$ and $x_f(t)$, respectively. This notation will be used throughout. In a more compact form the dynamics can be written as

$$\dot{x}(t) = -A(\mathcal{G})x(t) + Bu(t), \tag{3.1}$$

where $x(t) = [x_1(t), \dots, x_n(t)]^T \in \mathbb{R}^n$, $B = [b_1, \dots, b_n]^T \in \mathbb{R}^{n \times m}$, and the matrix representation of $\mathcal{G}$ is defined as $A(\mathcal{G}) := 2\Delta_s(\mathcal{G}) - \mathcal{A}(\mathcal{G})$, i.e.,

$$A(\mathcal{G}) = \begin{bmatrix} w_{11} & -w_{12} & \cdots & -w_{1n} \\ -w_{21} & w_{22} & -w_{23} & \vdots \\ \vdots & & \ddots & -w_{n-1,n} \\ -w_{n1} & \cdots & -w_{n,n-1} & w_{nn} \end{bmatrix}.$$

In this way, the matrix $A(\mathcal{G})$, as with the adjacency matrix, codifies the interconnections between nodes.

We define the class of matrices of this form as

$$Z_n = \left\{ A = (a_{ij}) \in \mathbb{R}^{n \times n} : a_{ii} \geq 0, a_{ij} \leq 0, i \neq j \right\}.$$

Z-matrices are matrices with nonpositive off-diagonals, hence Z_n is a subclass of Z-Matrices, motivating the name Z-Matrix dynamics for model (3.1).

The negation of Z-matrices falls into the class of *essentially nonnegative matrices*. These are matrices which have positive off-diagonals and bounded diagonals. This connection facilitates establishing positive invariance of the positive set $S^+ = \{x(t) | x(t) \geq 0\}$ with respect to model (3.1) when $A \in Z_n$. The result that establishes positive invariance and that will be used in the subsequent paper is as follows.

PROPOSITION 3.1. *Let A and B be essentially nonnegative matrices with $A \geq B$; that is $A + sI \geq 0$ and $B + sI \geq 0$ for all real s sufficiently large. For all $t \geq 0$, $e^{tA} \geq e^{tB} \geq 0$.*

Proof. Let s be sufficiently large such that $A + sI \geq 0$ and $B + sI \geq 0$, then $(A + sI)^j \geq (B + sI)^j \geq 0$ for all positive integer j. Hence,

$$e^{tB} = e^{t(B+sI)} = e^{-ts} \sum_{j=0}^{\infty} \frac{(t\,(B+sI))^j}{j!} \geq 0.$$

Further,

$$e^{tA} = e^{-ts} \sum_{j=0}^{\infty} \frac{(t\,(A+sI))^j}{j!} \geq e^{-ts} \sum_{j=0}^{\infty} \frac{(t\,(B+sI))^j}{j!} = e^{tB}. \qquad \square$$

We will now introduce families of dynamics with similar attributes via the construct of interval matrices.

4. Interval Matrices. We commence by defining interval matrices.

DEFINITION 4.1. *If $\underline{A}$ and $\overline{A}$ are two matrices in $\mathbb{R}^{n\times m}$ with $\underline{A} \leq \overline{A}$ then the set of matrices*

$$\mathbb{A} = \left[\underline{A}, \overline{A}\right] = \left\{A : \underline{A} \leq A \leq \overline{A}\right\}$$

is called an interval matrix, and the matrices $\overline{A}$ and $\underline{A}$ are called its bounds. Further, this interval is referred to as symmetric if $\overline{A}$ and $\underline{A}$ are symmetric.

Hence $A \in \mathbb{A}$ if $[\underline{A}]_{ij} \leq [A]_{ij} \leq [\overline{A}]_{ij}$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$. We emphasize that each entry in A can vary arbitrarily in its interval independent of the other entries in A. A symmetric interval matrix can also contain asymmetric matrices.

The examination of interval matrices arises naturally in control theory in connection with the behavior of linear time invariant systems under perturbations, and has been extensively studied. We refer the reader the the survey papers by Mansour [9] and Rohn [13] for a detailed list of references.

When the matrix intervals bounds are Z-matrices then all matrices contained in the interval are Z-matrices. A similar result is true for our subclass of matrix Z_n, this is stated formally without proof in the following proposition:

PROPOSITION 4.2. *Consider the matrices in the class Z_n, $A(\underline{\mathcal{G}})$ and $A(\overline{\mathcal{G}})$ with $A(\underline{\mathcal{G}}) \leq A(\overline{\mathcal{G}})$ corresponding to n-node graphs $\underline{\mathcal{G}}$ and $\overline{\mathcal{G}}$. Every matrix in the matrix interval $\mathbb{A} = \left[A(\underline{\mathcal{G}}), A(\overline{\mathcal{G}})\right]$ is in the class Z_n.*

The motivation for using matrix interval's is that Z-matrices exhibit many useful additive ordering properties and $A(\mathcal{G}) \in \left[A(\underline{\mathcal{G}}), A(\overline{\mathcal{G}})\right]$ implies that $\mathcal{G} \in \left[\underline{\mathcal{G}}, \overline{\mathcal{G}}\right]$ if and only if self-loops and their weights in $\mathcal{G}$, $\underline{\mathcal{G}}$ and $\overline{\mathcal{G}}$ are the same. The advantage of Z-matrix intervals is that trajectory bounds can be provided for dynamic systems defined by matrices contained in a Z-matrix interval matrix, as shown in the following:

PROPOSITION 4.3. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be finite graphs. Consider $\underline{x}(t)$ and $\overline{x}(t)$ to be the respective states of the systems*

$$\begin{aligned}\dot{\underline{x}}(t) &= -A(\underline{\mathcal{G}})\underline{x}(t) + \underline{Bu}(t)\\ \dot{\overline{x}}(t) &= -A(\overline{\mathcal{G}})\overline{x}(t) + \overline{B}\overline{u}(t).\end{aligned}$$

Then, the state trajectory generated by the dynamics

$$\dot{x}(t) = -Ax(t) + Bu(t),$$

for $A \in \mathbb{A} = \left[A(\underline{\mathcal{G}}), A(\overline{\mathcal{G}})\right]$, $B \in \mathbb{B} = \left[\overline{B}, \underline{B}\right]$ *and* $0 \leq \overline{u}(t) \leq u(t) \leq \underline{u}(t)$ *is bounded as*

$$\overline{x}(t) \leq x(t) \leq \underline{x}(t),$$

for all t *when initialized from* $0 \leq \overline{x}(0) \leq x(0) \leq \underline{x}(0)$.

Proof. For every $A \in \mathbb{A}$, $A(\underline{\mathcal{G}}) \leq A$. From Proposition 3.1, the unforced dynamics is bounded as

$$x_u(t) = e^{-At}x(0) \geq e^{-A(\underline{\mathcal{G}})t}x(0) \geq e^{-A(\underline{\mathcal{G}})t}\overline{x}(0) = \overline{x}_u(t).$$

Similarly, $x_u(t) \leq \underline{x}_u(t)$. For $t \geq \tau \geq 0$, from Proposition 3.1, the forced dynamics is bounded as

$$\begin{aligned} e^{-A(t-\tau)}Bu(\tau) &\geq e^{-A(\underline{\mathcal{G}})(t-\tau)}Bu(\tau) \\ &\geq e^{-A(\underline{\mathcal{G}})(t-\tau)}\overline{B}\overline{u}(\tau) \\ \int_0^t e^{-A(t-\tau)}Bu(\tau)d\tau &\geq \int_0^t e^{-A(\underline{\mathcal{G}})(t-\tau)}\overline{B}\overline{u}(\tau)d\tau \\ x_f(t) &\geq \overline{x}_f(t). \end{aligned}$$

Similarly, $x_f(t) \leq \underline{x}_f(t)$. Noting that the dynamics are formed by the sum of its unforced and forced dynamics, the proposition follows. □

The following section analyzes the Z-matrix dynamics and Z-matrix intervals dynamics formed from applying Cartesian products to graphs.

5. Z-Matrix Dynamics over Cartesian Products of Graphs. The Cartesian product over graphs can be formulated in terms of their Z-matrix representations using the Kronecker sum.

PROPOSITION 5.1. *Let* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *be a pair of graphs of order* n *and* m*, respectively. Then* $A(\mathcal{G}_1 \Box \mathcal{G}_2) = A(\mathcal{G}_1) \oplus A(\mathcal{G}_2)$.

Proof. The proposition follows directly from the definition of the graph product and the Z-matrix realization of a graph. □

We now present a result which we refer to as the *factorization lemma* for Z-matrix dynamics.

LEMMA 5.2. *[Factorization] Consider* $x_1(t)$ *and* $x_2(t)$ *to be the respective states of the systems*

$$\begin{aligned} \dot{x}_1(t) &= -A(\mathcal{G}_1)x_1(t) + B_1u_1(t) \\ \dot{x}_2(t) &= -A(\mathcal{G}_2)x_2(t) + B_2u_2(t), \end{aligned}$$

for all time t*. Then, the unforced state trajectory generated by the dynamics*

$$\dot{x}(t) = -A(\mathcal{G}_1 \Box \mathcal{G}_2)x(t) + (B_1 \otimes B_2)(u_1(t) \otimes u_2(t))$$

is

$$x_u(t) = x_{1u}(t) \otimes x_{2u}(t),$$

and the forced state trajectory is

$$x_f(t) = \int_0^t \dot{x}_{1f}(\tau) \otimes \dot{x}_{2f}(\tau)d\tau,$$

for all time t and with initial conditions $x(0) = x_1(0) \otimes x_2(0)$.

Proof. From Proposition 5.1, examining the unforced dynamics we have

$$\begin{aligned} x_u(t) &= e^{-A(\mathcal{G}_1 \Box \mathcal{G}_2)t} x(0) \\ &= e^{-A(\mathcal{G}_1)t \oplus -A(\mathcal{G}_2)t} (x_1(0) \otimes x_2(0)) \\ &= \left(e^{-A(\mathcal{G}_1)t} \otimes e^{-A(\mathcal{G}_2)t}\right)(x_1(0) \otimes x_2(0)) \\ &= e^{-A(\mathcal{G}_1)t} x_1(0) \otimes e^{-A(\mathcal{G}_2)t} x_2(0) \\ &= x_{1u}(t) \otimes x_{2u}(t). \end{aligned}$$

Examining the forced dynamics, we have

$$\begin{aligned} x_f(t) &= \int_0^t e^{-A(\mathcal{G}_1 \Box \mathcal{G}_2)(t-\tau)} (B_1 \otimes B_2)(u_1(\tau) \otimes u_2(\tau)) d\tau \\ &= \int_0^t \left(e^{-A(\mathcal{G}_1)(t-\tau)} \otimes e^{-A(\mathcal{G}_2)(t-\tau)}\right)(B_1 \otimes B_2)(u_1(\tau) \otimes u_2(\tau)) d\tau \\ &= \int_0^t z_1(\tau) \otimes z_2(\tau) d\tau, \end{aligned}$$

where $z_1(\tau) = e^{-A(\mathcal{G}_1)(t-\tau)} B_1 u_1(\tau)$ and $z_2(\tau) = e^{-A(\mathcal{G}_2)(t-\tau)} B_2 u_2(\tau)$. Noting that $\dot{x}_{if}(\tau) = \frac{d}{d\tau} \int_0^\tau z_i(\bar{t}) d\bar{t} = z_i(\tau)$, for $i = 1, 2$, the lemma follows. □

Figure 2.1 displays an example the unforced trajectories in Lemma 5.2. Note that, due to the associativity of the Cartesian product, the result extends to arbitrary chains of Cartesian products.

Observing that if $A(\underline{\mathcal{G}}_1) \leq A(\overline{\mathcal{G}}_1)$ and $A(\underline{\mathcal{G}}_2) \leq A(\overline{\mathcal{G}}_2)$ then $A(\underline{\mathcal{G}}_1 \Box \underline{\mathcal{G}}_2) \leq A(\overline{\mathcal{G}}_1 \Box \overline{\mathcal{G}}_2)$, we can define a Z-matrix interval $\left[A(\underline{\mathcal{G}}_1 \Box \underline{\mathcal{G}}_2), A(\overline{\mathcal{G}}_1 \Box \overline{\mathcal{G}}_2)\right]$ with composite interval matrix bounds. The following lemma shows that trajectories of dynamics defined by Z-matrices in this interval are bounded above and below by the trajectories of dynamics defined by the factor Z-matrices $A(\underline{\mathcal{G}}_1)$, $A(\underline{\mathcal{G}}_2)$, $A(\overline{\mathcal{G}}_1)$ and $A(\overline{\mathcal{G}}_2)$.

LEMMA 5.3. *[Interval Factorization] Consider* $\underline{x_1}(t), \overline{x}_1(t), \underline{x_2}(t)$ *and* $\overline{x}_2(t)$ *to be the respective states of the systems*

$$\begin{aligned} \dot{\underline{x}}_1(t) &= -A(\underline{\mathcal{G}}_1)\underline{x}_1(t) + \underline{B}_1 \underline{u}_1(t) \\ \dot{\overline{x}}_1(t) &= -A(\overline{\mathcal{G}}_1)\overline{x}_1(t) + \overline{B}_1 \overline{u}_1(t) \\ \dot{\underline{x}}_2(t) &= -A(\underline{\mathcal{G}}_2)\underline{x}_2(t) + \underline{B}_2 \underline{u}_2(t) \\ \dot{\overline{x}}_2(t) &= -A(\overline{\mathcal{G}}_2)\overline{x}_2(t) + \overline{B}_2 \overline{u}_2(t). \end{aligned}$$

Then, the unforced state trajectory generated by the dynamics

$$\dot{x}(t) = -Ax(t) + Bu(t),$$

for $A \in \mathbb{A} = \left[A(\underline{\mathcal{G}}_1 \Box \underline{\mathcal{G}}_2), A(\overline{\mathcal{G}}_1 \Box \overline{\mathcal{G}}_2)\right]$, $B \in \mathbb{B} = \left[\overline{B}_1 \otimes \overline{B}_2, \underline{B}_1 \otimes \underline{B}_2\right] \geq 0$, $\overline{u}_1(t) \otimes \overline{u}_2(t) \leq u(t) \leq \underline{u}_1(t) \otimes \underline{u}_2(t)$ *and* $0 \leq \overline{u}_i(t) \leq \underline{u}_i(t)$ *for* $i = 1, 2$ *is bounded as*

$$\overline{x}_{1u}(t) \otimes \overline{x}_{2u}(t) \leq x_u(t) \leq \underline{x}_{1u}(t) \otimes \underline{x}_{2u}(t),$$

and the forced state trajectory is bounded as

$$\int_0^t \dot{\overline{x}}_{1f}(\tau) \otimes \dot{\overline{x}}_{2f}(\tau) d\tau \leq x_f(t) \leq \int_0^t \dot{\underline{x}}_{1f}(\tau) \otimes \dot{\underline{x}}_{2f}(\tau) d\tau,$$

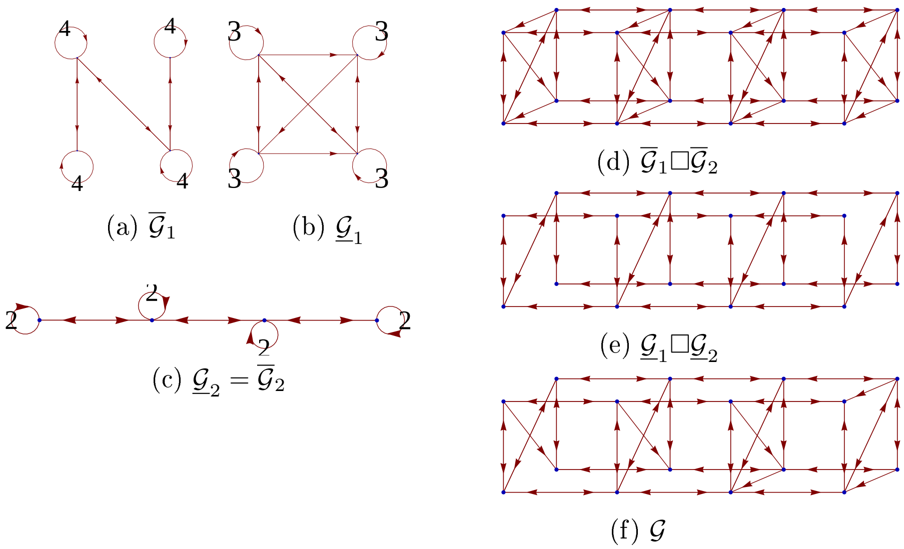

FIGURE 5.1. *The factor graphs $\overline{\mathcal{G}}_1$, $\underline{\mathcal{G}}_1$ and $\underline{\mathcal{G}}_2 = \overline{\mathcal{G}}_2$. The composite graphs $\overline{\mathcal{G}}_1 \square \overline{\mathcal{G}}_2$ and $\underline{\mathcal{G}}_1 \square \underline{\mathcal{G}}_2$ and graph $\mathcal{G}$ where $A(\mathcal{G}) \in \left[A(\underline{\mathcal{G}}_1 \square \underline{\mathcal{G}}_2), A(\overline{\mathcal{G}}_1 \square \overline{\mathcal{G}}_2)\right]$. For clarity, self-loops are unmarked for the graphs (d), (e) and (f) and have weight 5 in $\overline{\mathcal{G}}_1 \square \overline{\mathcal{G}}_2$ and $\mathcal{G}$ and weight 6 in $\underline{\mathcal{G}}_1 \square \underline{\mathcal{G}}_2$. All other edges have weight 1.*

for all time t with initial conditions

$$\overline{x}_1(0) \otimes \overline{x}_2(0) \leq x(0) \leq \underline{x}_1(0) \otimes \underline{x}_2(0),$$

and $0 \leq \overline{x}_i(t) \leq \underline{x}_i(t)$ for $i = 1, 2$.

Proof. Consider the dynamics

$$\begin{aligned}
\dot{\underline{z}}(t) &= -A(\underline{\mathcal{G}}_1 \square \underline{\mathcal{G}}_2)\underline{z}(t) + (\underline{B}_1 \otimes \underline{B}_2)\left(\underline{u}_1(t) \otimes \underline{u}_2(t)\right) \\
\dot{\overline{z}}(t) &= -A(\overline{\mathcal{G}}_1 \square \overline{\mathcal{G}}_2)\overline{z}(t) + \left(\overline{B}_1 \otimes \overline{B}_2\right)\left(\overline{u}_1(t) \otimes \overline{u}_2(t)\right) \\
\dot{x}(t) &= -Ax(t) + Bu(t)
\end{aligned}$$

where $0 \leq \overline{z}(0) \leq x(0) \leq \underline{z}(0)$. From Proposition 4.3, $\overline{z}(t) \leq x(t) \leq \underline{z}(t)$ for all $t > 0$. Further letting $\overline{z}(0) = \overline{x}_1(0) \otimes \overline{x}_2(0)$ and $\underline{z}(0) = \underline{x}_1(0) \otimes \underline{x}_2(0)$ using Lemma 5.2, the lemma follows. □

The significance of this result is that graph-based dynamics need not be composite to take advantage of the factorization lemma. Indeed its Z-matrix graph representation need only be bounded by the Z-matrix representations of its composite graphs. Further, the factor graph dynamics provides bounds on nonnegative composite dynamics independent of where the composite trajectory is initialized, i.e., $x(0)$ can be chosen anywhere in $(\mathbb{R}^{nm})_{\geq 0}$ rather than $\mathbb{R}^n \otimes \mathbb{R}^m \cong \mathbb{R}^{n+m}$ as for Lemma 5.2.

Figure 5.1 displays sample graphs $\underline{\mathcal{G}}_1$, $\overline{\mathcal{G}}_1$, $\underline{\mathcal{G}}_2$, $\overline{\mathcal{G}}_2$, $\underline{\mathcal{G}}_1 \square \underline{\mathcal{G}}_2$, $\overline{\mathcal{G}}_1 \square \overline{\mathcal{G}}_2$ and $\mathcal{G}$ pertaining to Lemma 5.3. The related trajectories of the 16 states are in Figure 5.2.

6. Conclusion. This paper presents an analysis for a class of dynamic networks involving Z-matrices. We explored the decomposition of such networks into smaller factor-networks. The trajectories of the composite network were generated from the factors' trajectories. Also, families of networks similar to the composite network were bounded by the factors' trajectories. Future work of particular interest is the examination of set controllability for such networks under graph products.

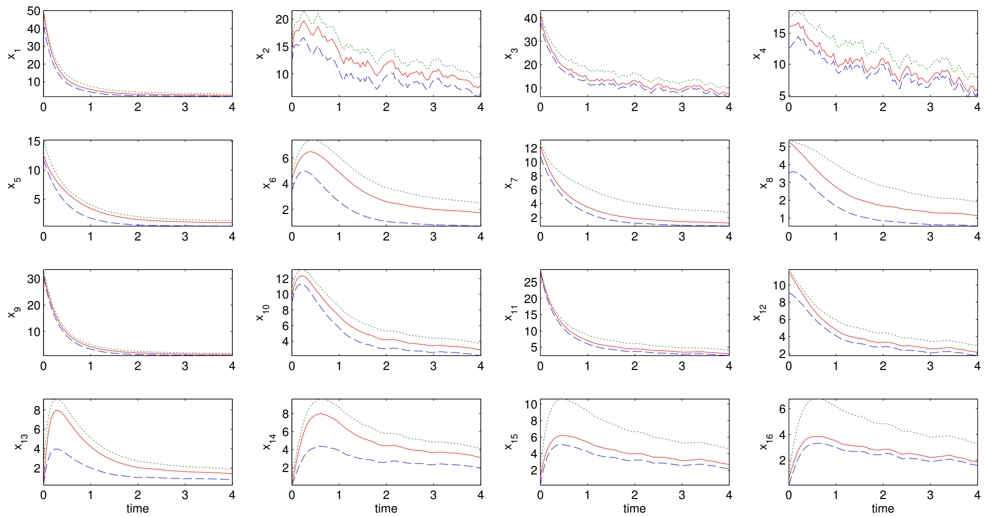

Figure 5.2. *Trajectories* $\overline{x}_{1u}(t)\otimes\overline{x}_{2u}(t)+\int_0^t \dot{\overline{x}}_{1f}(\tau)\otimes\dot{\overline{x}}_{2f}(\tau)d\tau$ *(blue/dashed),* $x(t)$ *(red/solid),* $\underline{x}_{1u}(t)\otimes\underline{x}_{2u}(t)+\int_0^t \dot{\underline{x}}_{1f}(\tau)\otimes\dot{\underline{x}}_{2f}(\tau)d\tau$ *(green/dotted) with underlying graph structure in Figure 5.1. The control matrices are* $\overline{B}_1=\underline{B}_1=e_1$, $\overline{B}_2=\underline{B}_2=[\boldsymbol{e}_2,\boldsymbol{e}_3+\boldsymbol{e}_4]$ *and* $B=\overline{B}_1\otimes\overline{B}_2$*. The controls are positive random signals satisfying the ordering requirement of Lemma 5.3.*

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